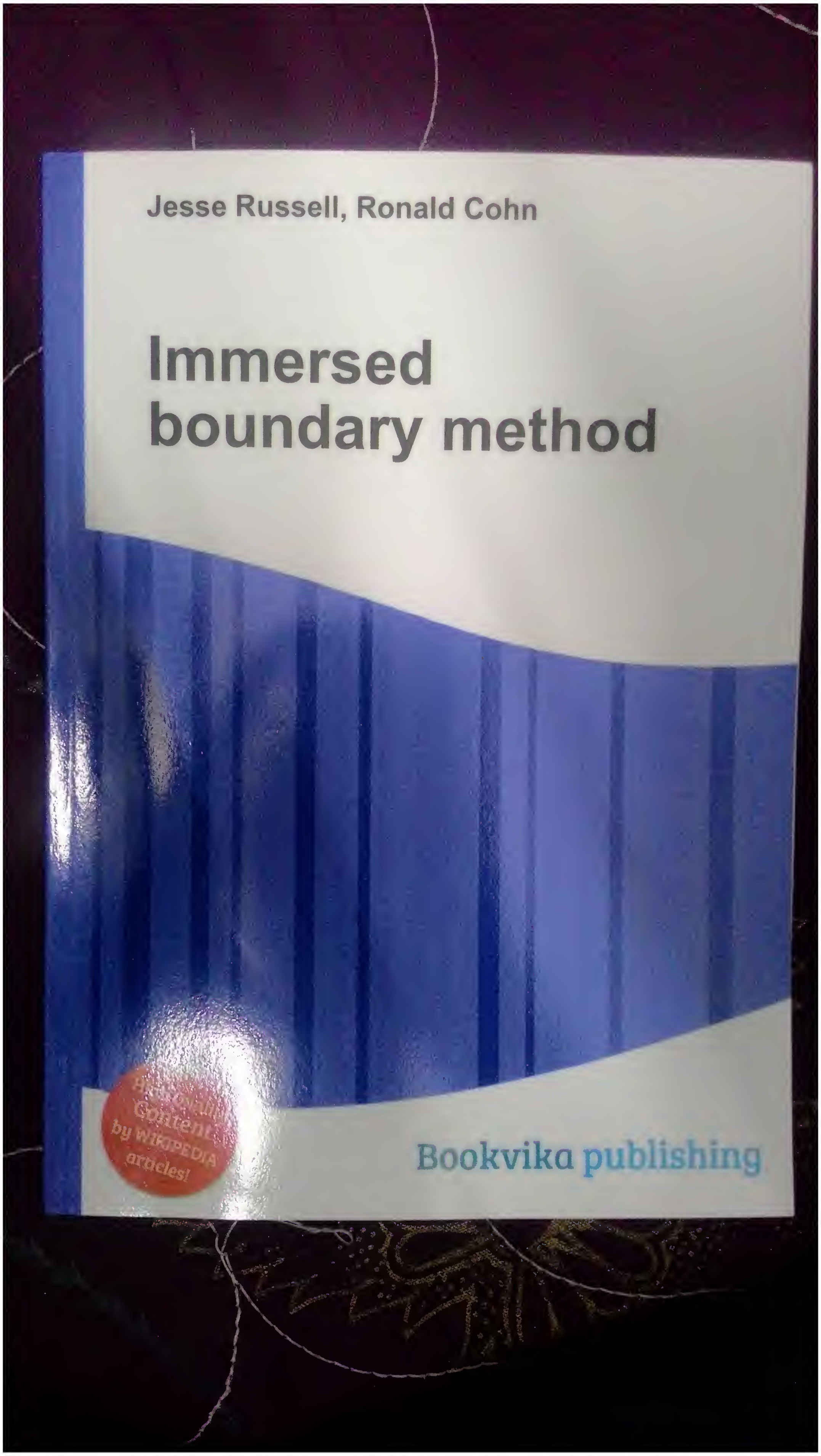
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Immersed boundary method
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Immersed boundary method

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Contents

Articles

References

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Immersed boundary method

The **immersed boundary method** is an approach – in computational fluid dynamics – to model and simulate mechanical systems in which elastic structures (or membranes) interact with fluid flows. Treating the coupling (the elastic boundary changes the flow of the fluid and the fluid moves the elastic boundary simultaneously) of the structure deformations and the fluid flow poses a number of challenging problems for numerical simulations. In the immersed boundary method approach the fluid is represented in an Eulerian coordinate frame and the structures in a Lagrangian coordinate frame. For Newtonian fluids governed by the Navier–Stokes equations the immersed boundary method fluid equations are

with incompressibility condition

The immersed structures are typically represented by a collection of interacting particles with a prescribed force law, where is the force acting on the particle. The forces are accounted for in the fluid equations by the force density

where is an approximation of the Dirac -function smoothed out over a length scale . The immersed structures are then updated using the equation

Variants of this basic approach have been applied to simulate a wide variety of mechanical systems involving elastic structures which interact with fluid flows. See the references for more details.

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8. Jindal S. et al. "The Immersed Boundary CFD Approach for Complex Aerodynamics Flow Predictions" SAE Journal, Detroit, Michigan 2007-01-0109 (2007)

Software : Numerical Codes

- Stochastic Immersed Boundary Methods in 3D, P. Atzberger, UCSB [2]
- Immersed Boundary Method for Uniform Meshes in 2D, A. Fogelson, Utah [3]
- Immersed Boundary Method for Adaptive Meshes in 3D, B. Griffith, NYU. [4]

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[4] http://www.math.nyu.edu/~griffith/IBAMR/

Stokesian dynamics

Stokesian dynamics[1] is a solution technique for the Langevin equation, which is the relevant form of Newton's 2nd law for a Brownian particle

In the above equation is the hydrodynamic force, i.e., force exerted by the fluid on the particle due to relative motion between them. is the stochastic Brownian force due to thermal motion of fluid particles. is the inter particle force,e.g. electrostatic repulsion between like charged particles. Brownian dynamics is one of the popular techniques of solving the Langevin equation, but the hydrodynamic interaction in Brownian dynamics is highly simplified and normally includes only the isolated body resistance. On the other hand, Stokesian dynamics includes the many body hydrodynamic interactions. Hydrodynamic interaction is very important for non-equilibrium suspensions, like a sheared suspension, where it plays a vital role in its microstructure and hence its properties. Stokesian dynamics is used primarily for non-equilibrium suspensions where it has been shown to provide results which agree with experiments.

Hydrodynamic interaction

One of the key features of Stokesian dynamics is its handing of the hydrodynamic interactions, which is fairly accurate without being computationally inhibitive (like boundary integral methods) for a large number of particles. Classical Stokesian dynamics requires operations where N is the number of particles in the system (usually a periodic box). Recent advances have reduced the computational cost to [2]

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Volume of fluid method


An illustration of fluid simulation using VOF method.

In computational fluid dynamics, the **volume of fluid method** (or in short **VOF method**) is a numerical technique for tracking and locating the free surface (or fluid-fluid interface). It belongs to the class of Eulerian methods which are characterized by a mesh that is either stationary or is moving in a certain prescribed manner to accommodate the evolving shape of the interface. As such, VOF is an advection scheme—a numerical recipe that allows the programmer to track the shape and position of the interface, but it is not a standalone flow solving algorithm. The Navier-Stokes equations describing the motion of the flow have to be solved separately. The same applies for all other advection algorithms.

History

The Volume of Fluid method is based on earlier Marker-and-Cell (MAC) methods. First accounts of what is now known as VOF have been given by Noh & Woodward (1976), where fraction function (see below) appeared, although first publication in a Journal was by Hirt & Nichols (1981). Since VOF method surpassed MAC by lowering computer storage requirements, it quickly became popular. Early applications include Torrey et al. from Los Alamos, who created VOF codes for NASA (1985,1987). First implementations of VOF suffered from imperfect interface description, which was later remedied by introducing a Piecewise-Linear Interface Calculation (PLIC) scheme .

Using VOF with PLIC is a contemporary standard, used in number of computer codes [1], including ANSYS Fluent.

Specification

The method is based on the idea of so-called fraction function . It is defined as the integral of fluid's characteristic function in the control volume (namely, volume of a computational grid cell). Basically, when the cell is empty, with no traced fluid inside. the value of is zero; when the cell is full, ; and when the interphasal interface cuts the cell, then . is a discontinuous function, its value jumps from 0 to 1 when the argument moves into interior of traced phase.

The fraction function is a scalar function, and while the fluid moves with velocity (in three-dimensional space) every fluid particle retains its identity, i.e. when a particle is a given phase, it doesn't change the phase - like a particle of air, that is a part of air bubble in water remains air particle, regardless of the bubble movement (actually. for this to hold, we have to disregard processes such as dissolving of air in water). If that is so, then the substantial derivative of fraction function needs to be equal to zero:

This is actually the same equation that has to be fulfilled by the level set distance function .

This equation cannot be easily solved directly, since is discontinuous, but such attempts have been performed. But the most popular approach to the equation is the so called geometrical reconstruction, originating in the works of Hirt and B. D. Nichols.

The VOF method is known for its ability to conserve the "mass" of the traced fluid, also, when fluid interface changes its topology, this change is traced easily, so the interfaces can for example join, or break apart

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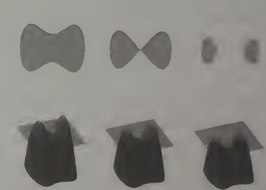
Level set method

The **level set method** (sometimes abbreviated as **LSM**) is a numerical technique for tracking interfaces and shapes. The advantage of the level set method is that one can perform numerical computations involving curves and surfaces on a fixed Cartesian grid without having to parameterize these objects (this is called the *Eulerian approach*).[1] Also, the level set method makes it very easy to follow shapes that change topology, for example when a shape splits in two, develops holes, or the reverse of these operations. All these make the level set method a great tool for modeling time-varying objects, like inflation of an airbag, or a drop of oil floating in water.

Level set method



An illustration of the level set method

A very simple, yet powerful way to understand the level set method is by first studying the accompanying illustration before proceeding towards a more technical definition, which then becomes quite accessible. The figure on the right illustrates several important ideas about the level set method. In the upper-left corner we see a shape, that is, a bounded region with a well-behaved boundary. Below it, the red surface is the graph of a level set function determining this shape, and the flat blue region represents the plane. The boundary of the shape is then the zero level set of , while the shape itself is the set of points in the plane for which is positive (interior of the shape) or zero (at the boundary).

In the top row we see the shape changing its topology by splitting in two. It would be quite hard to describe this transformation numerically

by parameterizing the boundary of the shape and following its evolution. One would need an algorithm able to detect the moment the shape splits in two, and then construct parameterizations for the two newly obtained curves. On the other hand, if we look at the bottom row, we see that the level set function merely got translated downward. We see that it is much easier to work with a shape through its level set function than with the shape directly, where we would need to watch out for all the possible deformations the shape might undergo.

Thus, in two dimensions, the level set method amounts to representing a closed curve (such as the shape in our example) using an auxiliary function , called the level set function. is represented as the zero level set of by

and the level set method manipulates *implicitly*, through the function . is assumed to take positive values inside the region delimited by the curve and negative values outside.[2][3]

The level set equation

If the curve moves in the normal direction with a speed , then the level set function satisfies the *level set equation*

Here, is the Euclidean norm (denoted customarily by single bars in PDEs), and is time. This is a partial differential equation, in particular a Hamilton-Jacobi equation, and can be solved numerically, for example by using finite differences on a Cartesian grid.[2][3]

The numerical solution of the level set equation, however, requires sophisticated techniques. Simple finite difference methods fail quickly. Upwinding methods, such as the Godunov method, fare better; however the level set method does not guarantee the conservation of the volume and the shape of the level set in an advection field that does conserve the shape and size, for example uniform or rotational velocity field. Instead, the shape of the level set may get severely distorted and the level set may vanish over several time steps. For this reason, high-order finite difference schemes are generally required, such as high-order essentially non-oscillatory (ENO) schemes, and even then, the feasibility of long-time simulations is questionable. Further sophisticated methods to deal with this difficulty have been developed, e.g., combinations of the level set method with tracing marker particles advected by the velocity field.[4]

Example

Consider a unit circle in , shrinking in on itself at a constant rate, i.e. each point on the boundary of the circle moves along its inwards pointing normal at some fixed speed. The circle will shrink and eventually collapse down to a point. If an initial distance field is constructed (i.e. a function whose value is the signed euclidean distance to the boundary, positive interior, negative exterior) on the initial circle, the normalised gradient of this field will be the circle normal.

If the field has a constant value subtracted from it in time, the zero level (which was the initial boundary) of the new fields will also be circular, and will similarly collapse to a point. This is due to this being effectively the temporal integration of the Eikonal equation with a fixed front velocity.

History

The level set method was developed in the 1980s by the American mathematicians Stanley Osher and James Sethian. It has become popular in many disciplines, such as image processing, computer graphics, computational geometry, optimization, and computational fluid dynamics.

A number of level set data structures have been developed to facilitate the use of the level set method in computer applications.

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External links

- See Ronald Fedkiw's academic web page (http://graphics.stanford.edu/~fedkiw/) for many stunning pictures and animations showing how the level set method can be used to model real life phenomena, like fire, water, cloth, fracturing materials, etc.
- Multivac (http://vivienmallet.net/fronts/) is a C++ library for front tracking in 2D with level set methods.
- James Sethian's web page (http://math.berkeley.edu/~sethian/) on level set method.
- Stanley Osher's homepage (http://www.math.ucla.edu/~sjo/)

Charles S. Peskin

Charles S. Peskin (born 15 April 1946)[1] is a professor of mathematics at the Courant Institute of Mathematical Sciences, New York University. He is a MacArthur Fellow, and a member of the National Academy of Science.

Peskin has been a leading worker in the area of mathematical biology and fluid dynamics, especially problems involving fluid-structure interactions. An especially significant contribution was his introduction of the Immersed Boundary Method to handle in a computionally tractable way the coupling between deformable immersed structures and fluid flows. This method has been applied in a variety of contexts including the study of blood flow in the heart, lift generation in insect flight, and wave propagation in the cochlea of the inner ear.

Peskin received his Ph.D. in physiology from Yeshiva University in 1972 and shortly thereafter joined the faculty of the Courant Institute. He has been a productive educator of applied mathematicians, and has advised 37 PhD students as of April 2009.

Awards

- George David Birkhoff Prize in Applied Mathematics from AMS–SIAM, 2003
- Mayer's Award for Excellence in Science and Technology, 1994
- Sidney Fernbach Award, Institute of Electrical and Electronics Engineers Computer Society, 1994
- Cray Research Information Technology Leadership Award for Breakthrough Computational Science, 1994
- Josiah Willard Gibbs Lecturer, American Mathematical Society, 1993
- New York University Margaret and Herman Sokol Faculty Award in the Sciences, 1992
- James H. Wilkinson Prize (SIAM) in Numerical Analysis and Scientific Computing, 1986[2]
- MacArthur Fellowship, 1983–1988

He has also been a member of the National Academy of Sciences since 1995.

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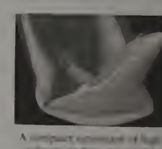
External links

- Professor Peskin's home page at NYU (http://www.math.nyu.edu/faculty/peskin/)

Computational fluid dynamics

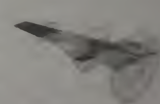
Computational fluid dynamics, usually abbreviated as **CFD**, is a branch of fluid mechanics that uses numerical methods and algorithms to solve and analyze problems that involve fluid flows. Computers are used to perform the calculations required to simulate the interaction of liquids and gases with surfaces defined by boundary conditions. With high-speed supercomputers, better solutions can be achieved. Ongoing research yields software that improves the accuracy and speed of complex simulation scenarios such as transonic or turbulent flows. Initial validation of such software is performed using a wind tunnel with the final validation coming in flight tests.

Background and history



A computer simulation of high velocity air flow around the Space Shuttle during re-entry.

The fundamental basis of almost all CFD problems are the Navier–Stokes equations, which define any single-phase fluid flow. These equations can be simplified by removing terms describing viscosity to yield the Euler equations. Further simplification, by removing terms describing vorticity yields the full potential equations. Finally, these equations can be linearized to yield the linearized potential equations.



A simulation of the Hyper-X scramjet vehicle in operation at Mach-7

Historically, methods were first developed to solve the Linearized Potential equations. Two-dimensional methods, using conformal transformations of the flow about a cylinder to the flow about an airfoil were developed in the 1930s.[1] The computer power available paced development of three-dimensional methods. The first paper on a practical three-dimensional method to solve the linearized potential equations was published by John Hess and A.M.O. Smith of Douglas Aircraft in 1967.[2] This method discretized the surface of the geometry with panels, giving rise to this class

of programs being called Panel Methods. Their method itself was simplified, in that it did not include lifting flows and hence was mainly applied to ship hulls and aircraft fuselages. The first lifting Panel Code (A230) was described in a paper written by Paul Rubbert and Gary Saaris of Boeing Aircraft in 1968. In time, more advanced three-dimensional Panel Codes were developed at Boeing (PANAIR, A502), Lockheed (Quadpan), Douglas (HESS), McDonnell Aircraft (MACAERO), NASA (PMARC) and Analytical Methods (WBAERO, USAERO and VSAERO). Some (PANAIR, HESS and MACAERO) were higher order codes, using higher order distributions of surface singularities, while others (Quadpan, PMARC, USAERO and VSAERO) used single singularities on each surface panel. The advantage of the lower order codes was that they ran much faster on the computers of the time. Today, VSAERO has grown to be a multi-order code and is the most widely used program of this class. It has been used in the development of many submarines, surface ships, automobiles, helicopters, aircraft, and more recently wind turbines. Its sister code, USAERO is an unsteady panel method that has also been used for modeling such things as high speed trains and racing yachts. The NASA PMARC code from an early version of VSAERO and a derivative of PMARC, named CMARC, is also commercially available.

In the two-dimensional realm, a number of Panel Codes have been developed for airfoil analysis and design. The codes typically have a boundary layer analysis included, so that viscous effects can be modeled. Professor Richard Eppler of the University of Stuttgart developed the PROFIL code, partly with NASA funding, which became available in the early 1980s. This was soon followed by MIT Professor Mark Drela's XFOIL code. Both PROFIL and XFOIL incorporate two-dimensional panel codes, with coupled boundary layer codes for airfoil analysis work. PROFIL uses a conformal transformation method for inverse airfoil design, while XFOIL has both a conformal transformation and an inverse panel method for airfoil design. Both codes are used.

An intermediate step between Panel Codes and Full Potential codes were codes that used the Transonic Small Disturbance equations. In particular, the three-dimensional WIBCO code, developed by Charlie Boppe of Grumman Aircraft in the early 1980s has seen heavy use.

Developers turned to Full Potential codes, as panel methods could not calculate the non-linear flow present at transonic speeds. The first description of a means of using the Full Potential equations was published by Earll Murman and Julian Cole of Boeing in 1970. Frances Bauer, Paul Garabedian and David Korn of the Courant Institute at New York University (NYU) wrote a series of two-dimensional Full Potential

airfoil codes that were widely used, the most important being named Program H. A further growth of Program H was developed by Bob Melnik and his group at Grumman Aerospace as Grumfoil. Antony Jameson, originally at Grumman Aircraft and the Courant Institute of NYU, worked with David Caughey to develop the important three-dimensional Full Potential code FLO22 in 1975. Many Full Potential codes emerged after this, culminating in Boeing's Tranair (A633) code, which still sees heavy use.

The next step was the Euler equations, which promised to provide more accurate solutions of transonic flows. The methodology used by Jameson in his three-dimensional FLO57 code (1981) was used by others to produce such programs as Lockheed's TEAM program and IAI/Analytical Methods' MGAERO program. MGAERO is unique in being a structured cartesian mesh code, while most other such codes use structured body-fitted grids (with the exception of NASA's highly successful CART3D code, Lockheed's SPLITFLOW code and Georgia Tech's NASCART-GT).[10] Antony Jameson also developed the three-dimensional AIRPLANE code (1985) which made use of unstructured tetrahedral grids.

In the two-dimensional realm, Mark Drela and Michael Giles, then graduate students at MIT, developed the ISES Euler program (actually a suite of programs) for airfoil design and analysis. This code first became available in 1986 and has been further developed to design, analyze and optimize single or multi-element airfoils, as the MSES program. MSES sees wide use throughout the world. A derivative of MSES, for the design and analysis of airfoils in a cascade, is MISES, developed by Harold "Guppy" Youngren while he was a graduate student at MIT.

The Navier–Stokes equations were the ultimate target of development. Two-dimensional codes, such as NASA Ames' ARC2D code first emerged. A number of three-dimensional codes were developed (ARC3D, OVERFLOW, CFL3D are three successful NASA contributions), leading to numerous commercial packages.

Methodology

In all of these approaches the same basic procedure is followed.

- During preprocessing
  - The geometry (physical bounds) of the problem is defined.
  - The volume occupied by the fluid is divided into discrete cells (the mesh). The mesh may be uniform or non uniform.
  - The physical modeling is defined – for example, the equations of motion + enthalpy + radiation + species conservation

- Boundary conditions are defined. This involves specifying the fluid behaviour and properties at the boundaries of the problem. For transient problems, the initial conditions are also defined.
- The simulation is started and the equations are solved iteratively as a steady-state or transient.
- Finally a postprocessor is used for the analysis and visualization of the resulting solution.

Discretization methods

The stability of the chosen discretization is generally established numerically rather than analytically as with simple linear problems. Special care must also be taken to ensure that the discretization handles discontinuous solutions gracefully. The Euler equations and Navier–Stokes equations both admit shocks, and contact surfaces.

Some of the discretization methods being used are:

Finite volume method

The finite volume method (FVM) is a common approach used in CFD codes. The governing equations are solved over discrete control volumes. Finite volume methods recast the governing partial differential equations (typically the Navier-Stokes equations) in a conservative form, and then discretize the new equation. This guarantees the conservation of fluxes through a particular control volume. The finite volume equation yields governing equations in the form,

where is the vector of conserved variables, is the vector of fluxes (see Euler equations or Navier–Stokes equations), is the volume of the control volume element, and is the surface area of the control volume element.

Finite element method

The finite element method (FEM) is used in structural analysis of solids, but is also applicable to fluids. However, the FEM formulation requires special care to ensure a conservative solution. The FEM formulation has been adapted for use with fluid dynamics governing equations. Although FEM must be carefully formulated to be conservative, it is much more stable than the finite volume approach[4] However, FEM can require more memory than FVM.[5]

In this method, a weighted residual equation is formed:

where is the equation residual at an element vertex , is the conservation equation expressed on an element basis, is the weight factor, and is the volume of the element.

Finite difference method

The finite difference method (FDM) has historical importance and is simple to program. It is currently only used in few specialized codes. Modern finite difference codes make use of an embedded boundary for handling complex geometries, making these codes highly efficient and accurate. Other ways to handle geometries include use of overlapping grids, where the solution is interpolated across each grid.

where is the vector of conserved variables, and , , and are the fluxes in the , , and directions respectively.

Spectral element method

Spectral element method is a finite element type method. It requires the mathematical problem (the partial differential equation) to be casted in a weak formulation. This is typically done by multiplying the differential equation by an arbitrary test function and integrating over the whole domain. Purely mathematically, the test functions are completely arbitrary - they belong to an infinitely dimensional function space. Clearly an infinitely dimensional function space cannot be represented on a discrete spectral element mesh. And this is where the spectral element discretization begins. The most crucial thing is the choice of interpolating and testing functions. In a standard, low order FEM in 2D, for quadrilateral elements the most typical choice is the bilinear test or interpolating function of the form . In a spectral element method however, the interpolating and test functions are chosen to be polynomials of a very high order (typically e.g. of the 10th order in CFD applications). This guarantees the rapid convergence of the method. Furthermore, very efficient integration procedures must be used, since the number of integrations to be performed in a numerical codes is big. Thus, high order Gauss integration quadratures are employed, since they achieve the highest accuracy with the smallest number of computations to be carried out. At the time there are some academic CFD codes based on the spectral element method and some more are currently under development, since the new time-stepping schemes arrise in the scientific world. You can refer to the C-CFD [6] website to see movies of incompressible flows in channels simulated with a spectral element solver or to the Numerical Mechanics (see bottom of the page) [7] website to see a movie of the lid-driven cavity flow obtained with a compeletely novel unconditionally stable time-stepping scheme combined with a spectral element solver.

Boundary element method

In the boundary element method, the boundary occupied by the fluid is divided into a surface mesh.

High-resolution discretization schemes

High-resolution schemes are used where shocks or discontinuities are present. Capturing sharp changes in the solution requires the use of second or higher-order numerical schemes that do not introduce spurious oscillations. This usually necessitates the application of flux limiters to ensure that the solution is total variation diminishing.

Turbulence models

In studying turbulent flows, the objective is to obtain a theory or a model that can yield quantities of interest, such as velocities. For turbulent flow, the range of length scales and complexity of phenomena make most approaches impossible. The primary approach in this case is to create numerical models to calculate the properties of interest. A selection of some commonly-used computational models for turbulent flows are presented in this section.

The chief difficulty in modeling turbulent flows comes from the wide range of length and time scales associated with turbulent flow. As a result, turbulence models can be classified based on the range of these length and time scales that are modeled and the range of length and time scales that are resolved. The more turbulent scales that are resolved, the finer the resolution of the simulation, and therefore the higher the computational cost. If a majority or all of the turbulent scales are modeled, the computational cost is very low, but the tradeoff comes in the form of decreased accuracy.

In addition to the wide range of length and time scales and the associated computational cost, the governing equations of fluid dynamics contain a non-linear convection term and a non-linear and non-local pressure gradient term. These nonlinear equations must be solved numerically with the appropriate boundary and initial conditions.

Reynolds-averaged Navier–Stokes

Reynolds-averaged Navier-Stokes (RANS) equations are the oldest approach to turbulence modeling. An ensemble version of the governing equations is solved, which introduces new *apparent stresses* known as Reynolds stresses. This adds a second order tensor of unknowns for which various models can provide different levels of closure. It is a common misconception that the RANS equations do not apply to flows

with a time-varying mean flow because these equations are 'time-averaged'. In fact, statistically unsteady (or non-stationary) flows can equally be treated. This is sometimes referred to as URANS. There is nothing inherent in Reynolds averaging to preclude this, but the turbulence models used to close the equations are valid only as long as the time over which these changes in the mean occur is large compared to the time scales of the turbulent motion containing most of the energy.

RANS models can be divided into two broad approaches:

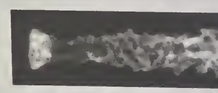
Boussinesq hypothesis

This method involves using an algebraic equation for the Reynolds stresses which include determining the turbulent viscosity, and depending on the level of sophistication of the model, solving transport equations for determining the turbulent kinetic energy and dissipation. Models include k-ε (Launder and Spalding),[8] Mixing Length Model (Prandtl),[9] and Zero Equation Model (Cebeci and Smith).[9] The models available in this approach are often referred to by the number of transport equations associated with the method. For example, the Mixing Length model is a "Zero Equation" model because no transport equations are solved; the is a "Two Equation" model because two transport equations (one for and one for) are solved.

Reynolds stress model (RSM)

This approach attempts to actually solve transport equations for the Reynolds stresses. This means introduction of several transport equations for all the Reynolds stresses and hence this approach is much more costly in CPU effort.

Large eddy simulation


Volume rendering of a non-premixed swirl flame as simulated by LES.

Large eddy simulation (LES) is a technique in which the smallest scales of the flow are removed through a filtering operation, and their effect modeled using subgrid scale models. This allows the largest and most important scales of the turbulence to be resolved, while greatly reducing the computational cost incurred by the smallest scales. This method requires greater computational resources than RANS methods, but is far cheaper than DNS.

Detached eddy simulation

Detached eddy simulations (DES) is a modification of a RANS model in which the model switches to a subgrid scale formulation in regions fine enough for LES calculations. Regions near solid boundaries and where the turbulent length scale is less than the maximum grid dimension are assigned the RANS mode of solution. As the turbulent length scale exceeds the grid dimension, the regions are solved using the LES mode. Therefore the grid resolution for DES is not as demanding as pure LES, thereby considerably cutting down the cost of the computation. Though DES was initially formulated for the Spalart-Allmaras model (Spalart et al., 1997), it can be implemented with other RANS models (Strelets, 2001), by appropriately modifying the length scale which is explicitly or implicitly involved in the RANS model. So while Spalart-Allmaras model based DES acts as LES with a wall model, DES based on other models (like two equation models) behave as a hybrid RANS-LES model. Grid generation is more complicated than for a simple RANS or LES case due to the RANS-LES switch. DES is a non-zonal approach and provides a single smooth velocity field across the RANS and the LES regions of the solutions.

Direct numerical simulation

Direct numerical simulation (DNS) resolves the entire range of turbulent length scales. This marginalizes the effect of models, but is extremely expensive. The computational cost is proportional to .[10] DNS is intractable for flows with complex geometries or flow configurations.

Coherent vortex simulation

The coherent vortex simulation approach decomposes the turbulent flow field into a coherent part, consisting of organized vortical motion, and the incoherent part, which is the random background flow.[11] This decomposition is done using wavelet filtering. The approach has much in common with LES, since it uses decomposition and resolves only the filtered portion, but different in that it does not use a linear, low-pass filter. Instead, the filtering operation is based on wavelets, and the filter can be adapted as the flow field evolves. Farge and Schneider tested the CVS method with two flow configurations and showed that the coherent portion of the flow exhibited the energy spectrum exhibited by the total flow, and corresponded to coherent structures (vortex tubes), while the incoherent parts of the flow composed homogeneous background noise, which exhibited no organized structures. Goldstein and Oleg[12] applied the CVS model to large eddy simulation, but did not assume that the wavelet filter completely eliminated all coherent motions from the

subfilter scales. By employing both LES and CVS filtering, they showed that the SFS dissipation was dominated by the SFS flow field's coherent portion.

PDF methods

Probability density function (PDF) methods for turbulence, first introduced by Lundgren,[13] are based on tracking the one-point PDF of the velocity, , which gives the probability of the velocity at point being between and . This approach is analogous to the kinetic theory of gases, in which the macroscopic properties of a gas are described by a large number of particles. PDF methods are unique in that they can be applied in the framework of a number of different turbulence models; the main differences occur in the form of the PDF transport equation. For example, in the context of large eddy simulation, the PDF becomes the filtered PDF.[14] PDF methods can also be used to describe chemical reactions,[15][16] and are particularly useful for simulating chemically reacting flows because the chemical source term is closed and does not require a model. The PDF is commonly tracked by using Lagrangian particle methods; when combined with large eddy simulation, this leads to a Langevin equation for subfiler particle evolution.

Vortex method

The vortex method is a grid-free technique for the simulation of turbulent flows. It uses vortices as the computational elements, mimicking the physical structures in turbulence. Vortex methods were developed as a grid-free methodology that would not be limited by the fundamental smoothing effects associated with grid-based methods. To be practical, however, vortex methods require means for rapidly computing velocities from the vortex elements – in other words they require the solution to a particular form of the N-body problem (in which the motion of N objects is tied to their mutual influences). A breakthrough came in the late 1980s with the development of the fast multipole method (FMM), an algorithm by V. Rokhlin (Yale) and L. Greengard (Courant Institute). This breakthrough paved the way to practical computation of the velocities from the vortex elements and is the basis of successful algorithms. They are especially well-suited to simulating filamentary motion, such as wisps of smoke, in real-time simulations such as video games, because of the fine detail achieved using minimal computation.[17]

Software based on the vortex method offer a new means for solving tough fluid dynamics problems with minimal user intervention. All that is required is specification of problem geometry and setting of boundary

and initial conditions. Among the significant advantages of this modern technology;

- It is practically grid-free, thus eliminating numerous iterations associated with RANS and LES.
- All problems are treated identically. No modeling or calibration inputs are required.
- Time-series simulations, which are crucial for correct analysis of acoustics, are possible.
- The small scale and large scale are accurately simulated at the same time.

Vorticity confinement method

The vorticity confinement (VC) method is an Eulerian technique used in the simulation of turbulent wakes. It uses a solitary-wave like approach to produce a stable solution with no numerical spreading. VC can capture the small scale features to within as few as 2 grid cells. Within these features, a nonlinear difference equation is solved as opposed to the finite difference equation. VC is similar to shock capturing methods, where conservation laws are satisfied, so that the essential integral quantities are accurately computed.

Two-phase flow

The modeling of two-phase flow is still under development. Different methods have been proposed . The Volume of fluid method has received a lot of attention lately, for problems that do not have dispersed particles, but the Level set method and front tracking are also valuable approaches . Most of these methods are either good in maintaining a sharp interface or at conserving mass . This is crucial since the evaluation of the density, viscosity and surface tension is based on the values averaged over the interface. Lagrangian multiphase models, which are used for dispersed media, are based on solving the Lagrangian equation of motion for the dispersed phase.

Solution algorithms

Discretization in space produces a system of ordinary differential equations for unsteady problems and algebraic equations for steady problems. Implicit or semi-implicit methods are generally used to integrate the ordinary differential equations, producing a system of (usually) nonlinear algebraic equations. Applying a Newton or Picard iteration produces a system of linear equations which is nonsymmetric in the presence of advection and indefinite in the presence of

incompressibility. Such systems, particularly in 3D, are frequently too large for direct solvers, so iterative methods are used, either stationary methods such as successive overrelaxation or Krylov subspace methods. Krylov methods such as GMRES, typically used with preconditioning, operate by minimizing the residual over successive subspaces generated by the preconditioned operator.

Multigrid has the advantage of asymptotically optimal performance on many problems. Traditional solvers and preconditioners are effective at reducing high-frequency components of the residual, but low-frequency components typically require many iterations to reduce. By operating on multiple scales, multigrid reduces all components of the residual by similar factors, leading to a mesh-independent number of iterations.

For indefinite systems, preconditioners such as incomplete LU factorization, additive Schwarz, and multigrid perform poorly or fail entirely, so the problem structure must be used for effective preconditioning.[18] Methods commonly used in CFD are the SIMPLE and Uzawa algorithms which exhibit mesh-dependent convergence rates, but recent advances based on block LU factorization combined with multigrid for the resulting definite systems have led to preconditioners that deliver mesh-independent convergence rates.[19]

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External links

- CFD Tutorial (http://www.societyofrobots.com/mechanics_FEA.shtml) Many examples and images, with references to robotic fish.
- CFD-Wiki (http://www.cfd-online.com/Wiki/Main_Page)
- *Course:* Introduction to CFD (http://www.mathematik.uni-dortmund.de/~kuzmin/cfdintro/cfd.html) – Dmitri Kuzmin (Dortmund University of Technology)
- CFD Success Stories (http://www.comsol.com/industry/application/chem/cfd/) Examples of Successful CFD simulation examples

Navier–Stokes equations

In physics, the **Navier–Stokes equations**, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the fluid stress is the sum of a diffusing viscous term (proportional to the gradient of velocity), plus a pressure term.

The equations are useful because they describe the physics of many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier–Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with Maxwell's equations they can be used to model and study magnetohydrodynamics.

The Navier–Stokes equations are also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, mathematicians have not yet proven that in three dimensions solutions always exist (existence), or that if they do exist, then they do not contain any singularity (smoothness). These are called the Navier–Stokes existence and smoothness problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example.[1]

The Navier–Stokes equations dictate not position but rather velocity. A solution of the Navier–Stokes equations is called a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. Once the velocity field is solved for, other quantities of interest (such as flow rate or drag force) may be found. This is different from what one normally sees in classical mechanics, where solutions are typically trajectories of position of a particle or deflection of a continuum. Studying velocity instead of position makes more sense for a fluid; however for visualization purposes one can compute various trajectories.

Properties

Nonlinearity

The Navier–Stokes equations are nonlinear partial differential equations in almost every real situation. In some cases, such as one-dimensional flow and Stokes flow (or creeping flow), the equations can be simplified to linear equations. The nonlinearity makes most problems difficult or impossible to solve and is the main contributor to the turbulence that the equations model.

The nonlinearity is due to convective acceleration, which is an acceleration associated with the change in velocity over position. Hence, any convective flow, whether turbulent or not, will involve nonlinearity. An example of convective but laminar (nonturbulent) flow would be the passage of a viscous fluid (for example, oil) through a small converging nozzle. Such flows, whether exactly solvable or not, can often be thoroughly studied and understood.

Turbulence

Turbulence is the time dependent chaotic behavior seen in many fluid flows. It is generally believed that it is due to the inertia of the fluid as a whole: the culmination of time dependent and convective acceleration; hence flows where inertial effects are small tend to be laminar (the Reynolds number quantifies how much the flow is affected by inertia). It is believed, though not known with certainty, that the Navier–Stokes equations describe turbulence properly.

The numerical solution of the Navier–Stokes equations for turbulent flow is extremely difficult, and due to the significantly different mixing-length scales that are involved in turbulent flow, the stable solution of this requires such a fine mesh resolution that the computational time becomes significantly infeasible for calculation (see Direct numerical simulation). Attempts to solve turbulent flow using a laminar solver typically result in a time-unsteady solution, which fails to converge appropriately. To counter this, time-averaged equations such as the Reynolds-averaged Navier–Stokes equations (RANS), supplemented with turbulence models, are used in practical computational fluid dynamics (CFD) applications when modeling turbulent flows. Some models include the Spalart-Allmaras, k-ω (k-omega), k-ε (k-epsilon), and SST models which add a variety of additional equations to bring closure to the RANS equations. Another technique for solving numerically the Navier–Stokes equation is the Large eddy simulation (LES). This approach is computationally more

expensive than the RANS method (in time and computer memory), but produces better results since the larger turbulent scales are explicitly resolved.

Applicability

Together with supplemental equations (for example, conservation of mass) and well formulated boundary conditions, the Navier–Stokes equations seem to model fluid motion accurately; even turbulent flows seem (on average) to agree with real world observations.

The Navier–Stokes equations assume that the fluid being studied is a continuum (it is infinitely divisible and not composed of particles such as atoms or molecules), and is not moving at relativistic velocities. At very small scales or under extreme conditions, real fluids made out of discrete molecules will produce results different from the continuous fluids modeled by the Navier–Stokes equations. Depending on the Knudsen number of the problem, statistical mechanics or possibly even molecular dynamics may be a more appropriate approach.

Another limitation is simply the complicated nature of the equations. Time tested formulations exist for common fluid families, but the application of the Navier–Stokes equations to less common families tends to result in very complicated formulations which are an area of current research. For this reason, these equations are usually written for Newtonian fluids. Studying such fluids is "simple" because the viscosity model ends up being linear; truly general models for the flow of other kinds of fluids (such as blood) do not, as of 2011, exist.

Derivation and description

The derivation of the Navier–Stokes equations begins with an application of Newton's second law: conservation of momentum (often alongside mass and energy conservation) being written for an arbitrary portion of the fluid. In an inertial frame of reference, the general form of the equations of fluid motion is:[2]

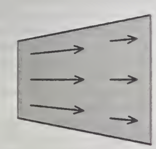
where is the flow velocity, is the fluid density, p is the pressure, is the (deviatoric) stress tensor, and represents body forces (per unit volume) acting on the fluid and is the del operator. This is a statement of the conservation of momentum in a fluid and it is an application of Newton's second law to a continuum; in fact this equation is applicable to any non-relativistic continuum and is known as the Cauchy momentum equation.

This equation is often written using the material derivative Dv/Dt, making it more apparent that this is a statement of Newton's second law:

The left side of the equation describes acceleration, and may be composed of time dependent or convective effects (also the effects of non-inertial coordinates if present). The right side of the equation is in effect a summation of body forces (such as gravity) and divergence of stress (pressure and shear stress).

Convective acceleration

A very significant feature of the Navier–Stokes equations is the presence of convective acceleration: the effect of time independent acceleration of a fluid with respect to space. While individual fluid particles are indeed experiencing time dependent acceleration, the convective acceleration of the flow field is a spatial effect, one example being fluid speeding up in a nozzle. Convective acceleration is represented by the nonlinear quantity:



An example of convection. Though the flow may be steady (time independent), the fluid decelerates as it moves down the diverging duct (assuming incompressible or subsonic compressible flow), hence there is an acceleration happening over position.

which may be interpreted either as or as with the tensor derivative of the velocity vector Both interpretations give the same result, independent of the coordinate system — provided is interpreted as the covariant derivative.[3]

Interpretation as $(\mathbf{v}\cdot\nabla)\mathbf{v}$

The convection term is often written as

where the advection operator is used. Usually this representation is preferred as it is simpler than the one in terms of the tensor derivative [3]

Interpretation as $\mathbf{v}\cdot(\nabla\mathbf{v})$

Here is the tensor derivative of the velocity vector, equal in Cartesian coordinates to the component by component gradient. The convection term may, by a vector calculus identity, be expressed without a tensor derivative:[4][5]

The form has use in irrotational flow, where the curl of the velocity (called vorticity) is equal to zero

Regardless of what kind of fluid is being dealt with, convective acceleration is a nonlinear effect. Convective acceleration is present in most flows (exceptions include one-dimensional incompressible flow), but its dynamic effect is disregarded in creeping flow (also called Stokes flow).

Stresses

The effect of stress in the fluid is represented by the and terms; these are gradients of surface forces, analogous to stresses in a solid. is called the pressure gradient and arises from the isotropic part of the stress tensor. This part is given by normal stresses that turn up in almost all situations, dynamic or not. The anisotropic part of the stress tensor gives rise to , which conventionally describes viscous forces; for incompressible flow, this is only a shear effect. Thus, is the deviatoric stress tensor, and the stress tensor is equal to:[6]

where is the 3x3 identity matrix. Interestingly, only the *gradient* of pressure matters, not the pressure itself. The effect of the pressure gradient is that fluid flows from high pressure to low pressure.

The stress terms p and are yet unknown, so the general form of the equations of motion is not usable to solve problems. Besides the equations of motion—Newton's second law—a force model is needed relating the stresses to the fluid motion.[7] For this reason, assumptions on the specific behavior of a fluid are made (based on natural observations) and applied in order to specify the stresses in terms of the other flow variables, such as velocity and density.

The Navier–Stokes equations result from the following assumptions on the deviatoric stress tensor :[8]

- the deviatoric stress vanishes for a fluid at rest, and – by Galilean invariance – also does not depend directly on the flow velocity itself, but only on spatial derivatives of the flow velocity
- in the Navier–Stokes equations, the deviatoric stress is expressed as the product of the tensor gradient of the flow velocity with a viscosity tensor , i.e. :
- the fluid is assumed to be isotropic, as valid for gases and simple liquids, and consequently is an isotropic tensor; furthermore, since the deviatoric stress tensor is symmetric, it turns out that it can be expressed in terms of two scalar dynamic viscosities μ and μ'': where is the rate-of-strain tensor and is the rate of expansion of the flow
- the deviatoric stress tensor has zero trace, so for a three-dimensional flow $2\mu + 3\mu'' = 0$

As a result, in the Navier–Stokes equations the deviatoric stress tensor has the following form:[8]

with the quantity between brackets the non-isotropic part of the rate-of-strain tensor The dynamic viscosity μ does not need to be constant – in general it depends on conditions like temperature and pressure, and in turbulence modelling the concept of eddy viscosity is used to approximate the average deviatoric stress.

The pressure p is modelled by use of an equation of state.[9] For the special case of an incompressible flow, the pressure constrains the flow in such a way that the volume of fluid elements is constant: isochoric flow resulting in a solenoidal velocity field with [10]

Other forces

The vector field represents body forces. Typically these consist of only gravity forces, but may include other types(such as electromagnetic forces). In a non-inertial coordinate system, other "forces" such as that associated with rotating coordinates may be inserted.

Often, these forces may be represented as the gradient of some scalar quantity , with . Gravity in the z direction, for example, is the gradient of . Since pressure shows up only as a gradient, this implies that solving a problem without any such body force can be mended to include the body force by using a modified pressure . The pressure and force terms on the right hand side of the Navier–Stokes equation become

Other equations

The Navier–Stokes equations are strictly a statement of the conservation of momentum. In order to fully describe fluid flow, more information is needed (how much depends on the assumptions made). This additional information may include boundary data (no-slip, capillary surface, etc.), the conservation of mass, the conservation of energy, and/or an equation of state.

Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation, given in its most general form as:

or, using the substantive derivative:

Incompressible flow of Newtonian fluids

A simplification of the resulting flow equations is obtained when considering an incompressible flow of a Newtonian fluid. The assumption of incompressibility rules out the possibility of sound or shock waves to occur; so this simplification is invalid if these phenomena are important. The incompressible flow assumption

typically holds well even when dealing with a "compressible" fluid — such as air at room temperature — at low Mach numbers (even when flowing up to about Mach 0.3). Taking the incompressible flow assumption into account and assuming constant viscosity, the Navier–Stokes equations will read, in vector form:[11]

Here **f** represents "other" body forces (forces per unit volume), such as gravity or centrifugal force. The shear stress term becomes the useful quantity (is the vector Laplacian) when the fluid is assumed incompressible, homogeneous and Newtonian, where is the (constant) dynamic viscosity.[12]

It's well worth observing the meaning of each term (compare to the Cauchy momentum equation):

Note that only the convective terms are nonlinear for incompressible Newtonian flow. The convective acceleration is an acceleration caused by a (possibly steady) change in velocity over *position*, for example the speeding up of fluid entering a converging nozzle. Though individual fluid particles are being accelerated and thus are under unsteady motion, the flow field (a velocity distribution) will not necessarily be time dependent.

Another important observation is that the viscosity is represented by the vector Laplacian of the velocity field (interpreted here as the difference between the velocity at a point and the mean velocity in a small volume around). This implies that Newtonian viscosity is **diffusion of momentum**, this works in much the same way as the diffusion of heat seen in the heat equation (which also involves the Laplacian).

If temperature effects are also neglected, the only "other" equation (apart from initial/boundary conditions) needed is the mass continuity equation. Under the incompressible assumption, density is a constant and it follows that the equation will simplify to:

This is more specifically a statement of the conservation of volume (see divergence).

These equations are commonly used in 3 coordinates systems: Cartesian, cylindrical, and spherical. While the Cartesian equations seem to follow directly from the vector equation above, the vector form of the Navier–Stokes equation involves some tensor calculus which means that writing it in other coordinate systems is not as simple as doing so for scalar equations (such as the heat equation).

Cartesian coordinates

Writing the vector equation explicitly,

Note that gravity has been accounted for as a body force, and the values of will depend on the orientation of gravity with respect to the chosen set of coordinates.

The continuity equation reads:

When the flow is at steady-state, does not change with respect to time. The continuity equation is reduced to:

When the flow is incompressible, is constant and does not change with respect to space. The continuity equation is reduced to:

The velocity components (the dependent variables to be solved for) are typically named u, v, w. This system of four equations comprises the most commonly used and studied form. Though comparatively more compact than other representations, this is still a nonlinear system of partial differential equations for which solutions are difficult to obtain.

Cylindrical coordinates

A change of variables on the Cartesian equations will yield[11] the following momentum equations for r, , and z:

The gravity components will generally not be constants, however for most applications either the coordinates are chosen so that the gravity components are constant or else it is assumed that gravity is counteracted by a pressure field (for example, flow in horizontal pipe is treated normally without gravity and without a vertical pressure gradient). The continuity equation is:

This cylindrical representation of the incompressible Navier–Stokes equations is the second most commonly seen (the first being Cartesian above). Cylindrical coordinates are chosen to take advantage of symmetry, so that a velocity component can disappear. A very common case is axisymmetric flow with the assumption of no tangential velocity (), and the remaining quantities are independent of :

Spherical coordinates

In spherical coordinates, the r, , and momentum equations are[11] (note the convention used: is polar angle, or colatitude,[13] $0 \leq \leq$):

Mass continuity will read:

These equations could be (slightly) compacted by, for example, factoring from the viscous terms. However, doing so would undesirably alter the structure of the Laplacian and other quantities.

Stream function formulation

Taking the curl of the Navier–Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed (and no dependence of anything on z), where the equations reduce to:

Differentiating the first with respect to y, the second with respect to x and subtracting the resulting equations will eliminate pressure and any conservative force. Defining the stream function through

results in mass continuity being unconditionally satisfied (given the stream function is continuous), and then incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

where is the (2D) biharmonic operator and is the kinematic viscosity, . We can also express this compactly using the Jacobian determinant:

This single equation together with appropriate boundary conditions describes 2D fluid flow, taking only kinematic viscosity as a parameter. Note that the equation for creeping flow results when the left side is assumed zero.

In axisymmetric flow another stream function formulation, called the Stokes stream function, can be used to describe the velocity components of an incompressible flow with one scalar function.

Pressure-free velocity formulation

The incompressible Navier–Stokes equation is a differential algebraic equation, having the inconvenient feature that there is no explicit mechanism for advancing the pressure in time. Consequently, much effort has been expended to eliminate the pressure from all or part of the computational process. The stream function formulation above eliminates the pressure (in 2D) at the expense of introducing higher derivatives and elimination of the velocity, which is the primary variable of interest.

The incompressible Navier–Stokes equation is composite, the sum of two orthogonal equations,

where and are solenoidal and irrotational projection operators satisfying and and are the nonconservative and conservative parts of the body force. This result follows from the Helmholtz Theorem (also known as the fundamental theorem of vector calculus). The first equation is a pressureless governing equation for the velocity, while the second equation for the pressure is a functional of the velocity and is related to the pressure Poisson equation.

The explicit functional form of the projection operator in 3D is found from the Helmholtz Theorem

with a similar structure in 2D. Thus the governing equation is an integro-differential equation and not convenient for numerical computation.

An equivalent weak or variational form of the equation, proved to produce the same velocity solution as the Navier–Stokes equation,[14] is given by,

for divergence-free test functions satisfying appropriate boundary conditions. Here, the projections are accomplished by the orthogonality of the solenoidal and irrotational function spaces. The discrete form of this is imminently suited to finite element computation of divergence-free flow, as we shall see in the next section. There we will be able to address the question, "How does one specify pressure-driven (Poiseuille) problems with a pressureless governing equation?"

The absence of pressure forces from the governing velocity equation demonstrates that the equation is not a dynamic one, but rather a kinematic equation where the divergence-free condition serves the role of a conservation law. This all would seem to refute the frequent statements that the incompressible pressure enforces the divergence-free condition.

Discrete velocity

With partitioning of the problem domain and defining basis functions on the partitioned domain, the discrete form of the governing equation is

It is desirable to choose basis functions which reflect the essential feature of incompressible flow – the elements must be divergence-free. While the velocity is the variable of interest, the existence of the stream function or vector potential is necessary by the Helmholtz Theorem. Further, to determine fluid flow in the absence of a pressure gradient, one can specify the difference of stream function values across a 2D channel, or the line integral of the tangential component of the vector potential around the channel in 3D, the flow being given by Stokes' Theorem. Discussion will be restricted to 2D in the following.

We further restrict discussion to continuous Hermite finite elements which have at least first-derivative degrees-of-freedom. With this, one can draw a large number of candidate triangular and rectangular elements from the plate-bending literature. These elements have derivatives as components of the gradient. In 2D, the gradient and curl of a scalar are clearly orthogonal, given by the expressions,

Adopting continuous plate-bending elements, interchanging the derivative degrees-of-freedom and changing the sign of the appropriate one gives many families of stream function elements.

Taking the curl of the scalar stream function elements gives divergence-free velocity elements[15],[16] The requirement that the stream function elements be continuous assures that the normal component of the velocity is continuous across element interfaces, all that is necessary for vanishing divergence on these interfaces.

Boundary conditions are simple to apply. The stream function is constant on no-flow surfaces, with no-slip velocity conditions on surfaces. Stream function differences across open channels determine the flow. No boundary conditions are necessary on open boundaries, though consistent values may be used with some problems. These are all Dirichlet conditions.

The algebraic equations to be solved are simple to set up, but of course are non-linear, requiring iteration of the linearized equations. .

Similar considerations apply to three-dimensions, but extension from 2D is not immediate because of the vector nature of the potential, and there exists no simple relation between the gradient and the curl as was the case in 2D.

Pressure recovery

Recovering pressure from the velocity field is easy. The discrete weak equation for the pressure gradient is,

where the test/weight functions are irrotational. Any conforming scalar finite element may be used. However, the pressure gradient field may also be of interest. In this case one can use scalar Hermite elements for the pressure. For the test/weight functions one would choose the irrotational vector elements obtained from the gradient of the pressure element.

Compressible flow of Newtonian fluids

There are some phenomena that are closely linked with fluid compressibility. One of the obvious examples is sound. Description of such phenomena requires more general presentation of the Navier–Stokes equation that takes into account fluid compressibility. If viscosity is assumed a constant, one additional term appears, as shown here:[17][18]

where is the volume viscosity coefficient, also known as *bulk viscosity*. This additional term disappears for an *incompressible fluid*, when the divergence of the flow equals zero.

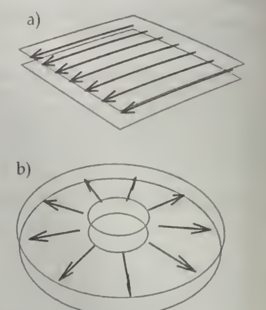
Application to specific problems

The Navier–Stokes equations, even when written explicitly for specific fluids, are rather generic in nature and their proper application to specific problems can be very diverse. This is partly because there is an enormous variety of problems that may be modeled, ranging from as simple as the distribution of static pressure to as complicated as multiphase flow driven by surface tension.

Generally, application to specific problems begins with some flow assumptions and initial/boundary condition formulation, this may be followed by scale analysis to further simplify the problem. For example, after assuming steady, parallel, one dimensional, nonconvective pressure driven flow between parallel plates, the resulting scaled (dimensionless) boundary value problem is:

The boundary condition is the no slip condition. This problem is easily solved for the flow field:

From this point onward more quantities of interest can be easily obtained, such as viscous drag force or net flow rate.



Visualization of a) parallel flow and b) radial flow.

Difficulties may arise when the problem becomes slightly more complicated. A seemingly modest twist on the parallel flow above would be the *radial* flow between parallel plates; this involves convection and thus nonlinearity. The velocity field may be represented by a function that must satisfy:

This ordinary differential equation is what is obtained when the Navier–Stokes equations are written and the flow assumptions applied (additionally, the pressure gradient is solved for). The nonlinear term makes this a very difficult problem to solve analytically (a lengthy implicit solution may be found which involves elliptic integrals and roots of cubic polynomials). Issues with the actual existence of solutions arise for R > 1.41 (approximately; this is not the square root of 2), the parameter R being the Reynolds number with appropriately chosen scales. This is an example of flow assumptions losing their applicability, and an example of the difficulty in "high" Reynolds number flows.

Exact solutions of the Navier–Stokes equations

Some exact solutions to the Navier–Stokes equations exist. Examples of degenerate cases — with the non-linear terms in the Navier–Stokes equations equal to zero — are Poiseuille flow, Couette flow and the oscillatory Stokes boundary layer. But also more interesting examples,

solutions to the full non-linear equations, exist; for example the Taylor–Green vortex.[19][20][21] Note that the existence of these exact solutions does not imply they are stable: turbulence may develop at higher Reynolds numbers.

A three dimensional steady-state vortex solution

A nice steady-state example with no singularities comes from considering the flow along the lines of a Hopf fibration. Let r be a constant radius to the inner coil. One set of solutions is given by[22]:

Some of the flow lines along a Hopf fibration.

for arbitrary constants A and B. This is a solution in a non-viscous gas (compressible fluid) whose density, velocities and pressure goes to zero far from the origin. (Note this is not a solution to the Clay Millennium problem because that refers to incompressible fluids where is a constant.) It is also worth pointing out that the components of the velocity vector are exactly those from the Pythagorean quadruple parametrization. Other choices of density and pressure are possible with the same velocity field:

Wyld diagrams

Wyld diagrams are bookkeeping graphs that correspond to the Navier–Stokes equations via a perturbation expansion of the fundamental continuum mechanics. Similar to the Feynman diagrams in quantum field theory, these diagrams are an extension of Keldysh's technique for nonequilibrium processes in fluid dynamics. In other words, these diagrams assign graphs to the (often) turbulent phenomena in turbulent fluids by allowing correlated and interacting fluid particles to obey stochastic processes associated to pseudo-random functions in probability distributions.[23]

Navier–Stokes equations use in games

The Navier–Stokes equations are used extensively in video games in order to model a wide variety of natural phenomena. These include simulations of effects such as water, fire, smoke etc. Many of the implementations used are based on the seminal paper "Real-Time Fluid Dynamics for Games"[24] by J. Stam. More recent implementations based upon this work run on the GPU as opposed to the CPU and achieve a much higher degree of performance.[25] [26]

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External links

- Simplified derivation of the Navier–Stokes equations (http://www.allstar.fiu.edu/aero/Flow2.htm)
- Millennium Prize problem description. (http://www.claymath.org/millennium/Navier-Stokes_Equations/navierstokes.pdf)
- CFD online software list (http://www.cfd-online.com/Wiki/Codes) A compilation of codes, including Navier–Stokes solvers.

Newtonian fluid

A **Newtonian fluid** (named after Isaac Newton) is a fluid whose stress versus strain rate curve is linear and passes through the origin.[1] [2][3] The constant of proportionality is known as the viscosity.

Definition

A simple equation to describe Newtonian fluid behaviour is

where

is the shear stress exerted by the fluid ("drag") [Pa]

is the fluid viscosity - a constant of proportionality [Pa·s]

is the velocity gradient perpendicular to the direction of shear, or equivalently the strain rate [s^{-1}]

In common terms, this means the fluid continues to flow, regardless of the forces acting on it. For example, water is Newtonian, because it continues to exemplify fluid properties no matter how fast it is stirred or mixed. Other examples may be aqueous solutions, emulsions. Contrast this with a non-Newtonian fluid, in which stirring can either leave a "hole" behind (that gradually fills up over time - this behavior is seen in materials such as pudding and oobleck, or, to a less rigorous extent, sand), or climb the stirring rod (the Weissenberg effect) because of shear thinning, the drop in viscosity causing it to flow more (this is seen in non-drip paints, which brush on easily but become more viscous when on walls).

For a Newtonian fluid, the viscosity, by definition, depends only on temperature and pressure (and also the chemical composition of the fluid if the fluid is not a pure substance), not on the forces acting upon it.

If the fluid is incompressible and viscosity is constant across the fluid, the equation governing the shear stress, in the Cartesian coordinate system, is the tensor

A second tensor, (also written as), representing the total stress, can be written by combining the shear stress with conventional (thermodynamic) pressure:

where, by the convention of tensor notation,

is the shear stress on the th face of a fluid element in the th direction

is thermodynamical pressure

is the velocity in the th direction

is the th direction coordinate

If a fluid does not obey this relation, it is termed a non-Newtonian fluid, of which there are several types, including polymer solutions, molten polymers, many solid suspensions and most highly viscous fluids.

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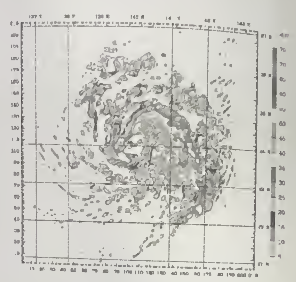
Computer simulation



A 48 hour computer simulation of Typhoon Mawar using the Weather Research and Forecasting model

A **computer simulation**, a **computer model**, or a **computational model** is a computer program, or network of computers, that attempts to simulate an abstract model of a particular system. Computer simulations have become a useful part of mathematical modeling of many natural systems in physics (computational physics), astrophysics, chemistry and biology, human systems in economics, psychology, social science, and engineering. Simulations can be used to explore and gain new insights into new technology, and to estimate the performance of systems too complex for analytical solutions.[1]

Computer simulations vary from computer programs that run a few minutes, to network-based groups of computers running for hours, to ongoing simulations that run for days. The scale of events being simulated by computer simulations has far exceeded anything possible (or perhaps even imaginable) using traditional paper-and-pencil mathematical modeling. Over 10 years ago, a desert-battle simulation, of one force invading another, involved the modeling of 66,239 tanks, trucks and other vehicles on simulated terrain around Kuwait, using multiple supercomputers in the DoD High Performance Computer Modernization Program[2] Other examples include a 1-billion-atom model of material deformation (2002) [3]; a 2.64-million-atom model of the complex maker of protein in all organisms, a ribosome, in 2005;[4] and the Blue Brain project at EPFL (Switzerland), begun in May 2005, to create the first computer simulation of the entire human brain, right down to the molecular level.[5]

Simulation versus modeling

Traditionally, building large models of systems has been via a statistical model, which attempts to find analytical solutions to problems and thereby enable the prediction of the behavior of the system from a set of parameters and initial conditions.

The term *computer simulation* is broader than *computer modeling*; the latter implies that all aspects are being modeled in the computer representation. However, computer simulation also includes generating inputs from simulated users in order to run actual computer software or equipment, with only part of the system being modeled. An example would be a flight simulator that can run machines as well as actual flight software.

Computer simulations are used in many fields, including science, technology, entertainment, health care, and business planning and scheduling.

History

Computer simulation developed hand-in-hand with the rapid growth of the computer, following its first large-scale deployment during the Manhattan Project in World War II to model the process of nuclear detonation. It was a simulation of 12 hard spheres using a Monte Carlo algorithm. Computer simulation is often used as an adjunct to, or substitute for, modeling systems for which simple closed form analytic solutions are not possible. There are many types of computer simulations; the common feature they all share is the attempt to generate a sample of representative scenarios for a model in which a complete enumeration of all possible states of the model would be prohibitive or impossible.

Data preparation

The external data requirements of simulations and models vary widely. For some, the input might be just a few numbers (for example, simulation of a waveform of AC electricity on a wire), while others might require terabytes of information (such as weather and climate models).

Input sources also vary widely:

- Sensors and other physical devices connected to the model;
- Control surfaces used to direct the progress of the simulation in some way;

- Current or Historical data entered by hand;
- Values extracted as by-product from other processes;
- Values output for the purpose by other simulations, models, or processes.

Lastly, the time at which data is available varies:

- "invariant" data is often built into the model code, either because the value is truly invariant (e.g. the value of π) or because the designers consider the value to be invariant for all cases of interest;
- data can entered into the simulation when it starts up, for example by reading one or more files, or by reading data from a preprocessor;
- data can be provided during the simulation run, for example by a sensor network;

Because of this variety, and that many common elements exist between diverse simulation systems, there are a large number of specialized simulation languages. The best-known of these may be Simula (sometimes Simula-67, after the year 1967 when it was proposed). There are now many others.

Systems that accept data from external sources must be very careful in knowing what they are receiving. While it is easy for computers to read in values from text or binary files, what is much harder is knowing what the accuracy (compared to measurement resolution and precision) of the values is. Often it is expressed as "error bars", a minimum and maximum deviation from the value seen within which the true value (is expected to) lie. Because digital computer mathematics is not perfect, rounding and truncation errors will multiply this error up, and it is therefore useful to perform an "error analysis"[6] to check that values output by the simulation are still usefully accurate.

Even small errors in the original data can accumulate into substantial error later in the simulation. While all computer analysis is subject to the "GIGO" (garbage in, garbage out) restriction, this is especially true of digital simulation. Indeed, it was the observation of this inherent, cumulative error, for digital systems that is the origin of chaos theory.

Types

Computer models can be classified according to several independent pairs of attributes, including:

- Stochastic or deterministic (and as a special case of deterministic, chaotic) - see External links below for examples of stochastic vs. deterministic simulations
- Steady-state or dynamic

- Continuous or discrete (and as an important special case of discrete, discrete event or DE models)
- Local or distributed.

Another way of categorizing models is to look at the underlying data structures. For time-stepped simulations, there are two main classes:

- Simulations which store their data in regular grids and require only next-neighbor access are called stencil codes. Many CFD applications belong to this category.
- If the underlying graph is not a regular grid, the model may belong to the meshfree method class.

Equations define the relationships between elements of the modeled system and attempt to find a state in which the system is in equilibrium. Such models are often used in simulating physical systems, as a simpler modeling case before dynamic simulation is attempted.

- Dynamic simulations model changes in a system in response to (usually changing) input signals.
- *Stochastic* models use *random number generators* to model chance or random events;
- A *discrete event simulation* (DES) manages events in time. Most computer, logic-test and fault-tree simulations are of this type. In this type of simulation, the simulator maintains a queue of events sorted by the simulated time they should occur. The simulator reads the queue and triggers new events as each event is processed. It is not important to execute the simulation in real time. It's often more important to be able to access the data produced by the simulation, to discover logic defects in the design, or the sequence of events.
- A *continuous dynamic simulation* performs numerical solution of differential-algebraic equations or differential equations (either partial or ordinary). Periodically, the simulation program solves all the equations, and uses the numbers to change the state and output of the simulation. Applications include flight simulators, construction and management simulation games, chemical process modeling, and simulations of electrical circuits. Originally, these kinds of simulations were actually implemented on analog computers, where the differential equations could be represented directly by various electrical components such as op-amps. By the late 1980s, however, most "analog" simulations were run on conventional digital computers that emulate the behavior of an analog computer.
- A special type of discrete simulation that does not rely on a model with an underlying equation, but can nonetheless be represented formally, is *agent-based simulation*. In agent-based simulation, the individual entities (such as molecules, cells, trees or consumers) in the model are represented directly (rather than by their density or

concentration) and possess an internal *state* and set of behaviors or *rules* that determine how the agent's state is updated from one time-step to the next.

- Distributed models run on a network of interconnected computers, possibly through the Internet. Simulations dispersed across multiple host computers like this are often referred to as "distributed simulations". There are several standards for distributed simulation, including Aggregate Level Simulation Protocol (ALSP), Distributed Interactive Simulation (DIS), the High Level Architecture (simulation) (HLA) and the Test and Training Enabling Architecture (TENA).

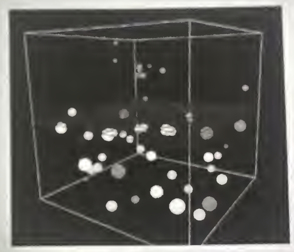
CGI computer simulation

Formerly, the output data from a computer simulation was sometimes presented in a table, or a matrix, showing how data was affected by numerous changes in the simulation parameters. The use of the matrix format was related to traditional use of the matrix concept in mathematical models; however, psychologists and others noted that humans could quickly perceive trends by looking at graphs or even moving-images or motion-pictures generated from the data, as displayed by computer-generated-imagery (CGI) animation. Although observers couldn't necessarily read out numbers, or spout math formulas, from observing a moving weather chart, they might be able to predict events (and "see that rain was headed their way"), much faster than scanning tables of rain-cloud coordinates. Such intense graphical displays, which transcended the World of numbers and formulae, sometimes also led to output that lacked a coordinate grid or omitted timestamps, as if straying too far from numeric data displays. Today, weather forecasting models tend to balance the view of moving rain/snow clouds against a map that uses numeric coordinates and numeric timestamps of events.
Similarly, CGI computer simulations of CAT scans can simulate how a tumor might shrink or change, during an extended period of medical treatment, presenting the passage of time as a spinning view of the visible human head, as the tumor changes.

Other applications of CGI computer simulations are being developed to graphically display large amounts of data, in motion, as changes occur during a simulation run.

Computer simulation in science

Generic examples of types of computer simulations in science, which are derived from an underlying mathematical description:



Computer simulation of the process of osmosis

- a numerical simulation of differential equations that cannot be solved analytically, theories that involve continuous systems such as phenomena in physical cosmology, fluid dynamics (e.g. climate models, roadway noise models, roadway air dispersion models), continuum mechanics and chemical kinetics fall into this category.
- a stochastic simulation, typically used for discrete systems where events occur probabilistically, and which cannot be described directly with differential equations (this is a *discrete* simulation in the above sense). Phenomena in this category include genetic drift, biochemical or gene regulatory networks with small numbers of molecules. (see also: Monte Carlo method).

Specific examples of computer simulations follow:

- statistical simulations based upon an agglomeration of a large number of input profiles, such as the forecasting of equilibrium temperature of receiving waters, allowing the gamut of meteorological data to be input for a specific locale. This technique was developed for thermal pollution forecasting .
- agent based simulation has been used effectively in ecology, where it is often called *individual based modeling* and has been used in situations for which individual variability in the agents cannot be neglected, such as population dynamics of salmon and trout (most purely mathematical models assume all trout behave identically).

- time stepped dynamic model. In hydrology there are several such hydrology transport models such as the SWMM and DSSAM Models developed by the U.S. Environmental Protection Agency for river water quality forecasting.
- computer simulations have also been used to formally model theories of human cognition and performance, e.g. ACT-R
- computer simulation using molecular modeling for drug discovery
- computer simulation for studying the selective sensitivity of bonds by mechanochemistry during grinding of organic molecules.[7]
- Computational fluid dynamics simulations are used to simulate the behaviour of flowing air, water and other fluids. There are one-, two- and three- dimensional models used. A one dimensional model might simulate the effects of water hammer in a pipe. A two-dimensional model might be used to simulate the drag forces on the cross-section of an aeroplane wing. A three-dimensional simulation might estimate the heating and cooling requirements of a large building.
- An understanding of statistical thermodynamic molecular theory is fundamental to the appreciation of molecular solutions. Development of the Potential Distribution Theorem (PDT) allows one to simplify this complex subject to down-to-earth presentations of molecular theory.

Notable, and sometimes controversial, computer simulations used in science include: Donella Meadows' World3 used in the *Limits to Growth*, James Lovelock's Daisyworld and Thomas Ray's Tierra.

Simulation environments for physics and engineering

Graphical environments to design simulations have been developed. Special care was taken to handle events (situations in which the simulation equations are not valid and have to be changed). The open project Open Source Physics was started to develop reusable libraries for simulations in Java, together with Easy Java Simulations, a complete graphical environment that generates code based on these libraries.

Computer simulation in practical contexts

Computer simulations are used in a wide variety of practical contexts, such as:

- analysis of air pollutant dispersion using atmospheric dispersion modeling
- design of complex systems such as aircraft and also logistics systems.
- design of Noise barriers to effect roadway noise mitigation
- flight simulators to train pilots
- weather forecasting
- Simulation of other computers is emulation.
- forecasting of prices on financial markets (for example Adaptive Modeler)
- behavior of structures (such as buildings and industrial parts) under stress and other conditions
- design of industrial processes, such as chemical processing plants
- Strategic Management and Organizational Studies
- Reservoir simulation for the petroleum engineering to model the subsurface reservoir
- Process Engineering Simulation tools.
- Robot simulators for the design of robots and robot control algorithms
- Urban Simulation Models that simulate dynamic patterns of urban development and responses to urban land use and transportation policies. See a more detailed article on Urban Environment Simulation.
- Traffic engineering to plan or redesign parts of the street network from single junctions over cities to a national highway network, for transportation system planning, design and operations. See a more detailed article on Simulation in Transportation.
- modeling car crashes to test safety mechanisms in new vehicle models

The reliability and the trust people put in computer simulations depends on the validity of the simulation model, therefore verification and validation are of crucial importance in the development of computer simulations. Another important aspect of computer simulations is that of reproducibility of the results, meaning that a simulation model should not provide a different answer for each execution. Although this might seem obvious, this is a special point of attention in stochastic simulations, where random numbers should actually be semi-random numbers. An exception to reproducibility are human in the loop simulations such as flight simulations and computer games. Here a human is part of the simulation and thus influences the outcome in a way that is hard, if not impossible, to reproduce exactly.

Vehicle manufacturers make use of computer simulation to test safety features in new designs. By building a copy of the car in a physics simulation environment, they can save the hundreds of thousands of dollars that would otherwise be required to build a unique prototype and test it. Engineers can step through the simulation milliseconds at a time to determine the exact stresses being put upon each section of the prototype.[8]

Computer graphics can be used to display the results of a computer simulation. Animations can be used to experience a simulation in real-time e.g. in training simulations. In some cases animations may also be useful in faster than real-time or even slower than real-time modes. For example, faster than real-time animations can be useful in visualizing the buildup of queues in the simulation of humans evacuating a building. Furthermore, simulation results are often aggregated into static images using various ways of scientific visualization.

In debugging, simulating a program execution under test (rather than executing natively) can detect far more errors than the hardware itself can detect and, at the same time, log useful debugging information such as instruction trace, memory alterations and instruction counts. This technique can also detect buffer overflow and similar "hard to detect" errors as well as produce performance information and tuning data.

Pitfalls

Although sometimes ignored in computer simulations, it is very important to perform sensitivity analysis to ensure that the accuracy of the results are properly understood. For example, the probabilistic risk analysis of factors determining the success of an oilfield exploration program involves combining samples from a variety of statistical distributions using the Monte Carlo method. If, for instance, one of the key parameters (e.g. the net ratio of oil-bearing strata) is known to only one significant figure, then the result of the simulation might not be more precise than one significant figure, although it might (misleadingly) be presented as having four significant figures.

Model Calibration Techniques

The following three steps should be used to produce accurate simulation models: calibration, verification, and validation. Computer simulations are good at portraying and comparing theoretical scenarios but in order to accurately model actual case studies, it has to match what is actually happening today. A base model should be created and calibrated so that it matches the area being studied. The calibrated model should then be

verified to ensure that the model is operating as expected based on the inputs. Once the model has been verified, the final step is to validate the model by comparing the outputs to historical data from the study area. This can be done by using statistical techniques and ensuring an adequate R-squared value. Unless these techniques are employed, the simulation model created will produce inaccurate results and not be a useful prediction tool.

Model calibration is achieved by adjusting any available parameters in order to adjust how the model operates and simulates the process. For example in traffic simulation, typical parameters include look-ahead distance, car-following sensitivity, discharge headway, and start-up lost time. These parameters influence driver behaviors such as when and how long it takes a driver to change lanes, how much distance a driver leaves between itself and the car in front of it, and how quickly it starts to accelerate through an intersection. Adjusting these parameters has a direct effect on the amount of traffic volume that can traverse through the modeled roadway network by making the drivers more or less aggressive. These are examples of calibration parameters that can be fine-tuned to match up with characteristics observed in the field at the study location. Most traffic models will have typical default values but they may need to be adjusted to better match the driver behavior at the location being studied.

Model verification is achieved by obtaining output data from the model and comparing it to what is expected from the input data. For example in traffic simulation, traffic volume can be verified to ensure that actual volume throughput in the model is reasonably close to traffic volumes input into the model. Ten percent is a typical threshold used in traffic simulation to determine if output volumes are reasonably close to input volumes. Simulation models handle model inputs in different ways so traffic that enters the network, for example, may or may not reach its desired destination. Additionally, traffic that wants to enter the network may not be able to, if any congestion exists. This is why model verification is a very important part of the modeling process.

The final step is to validate the model by comparing the results with what's expected based on historical data from the study area. Ideally, the model should produce similar results to what has happened historically. This is typically verified by nothing more than quoting the R2 statistic from the fit. This statistic measures the fraction of variability that is accounted for by the model. A high R2 value does not necessarily mean the model fits the data well. Another tool used to validate models is graphical residual analysis. If model output values are drastically different than historical values, it probably means there's an error in the model. This is an important step to verify before using the model as a

base to produce additional models for different scenarios to ensure each one is accurate. If the outputs do not reasonably match historic values during the validation process, the model should be reviewed and updated to produce results more in line with expectations. It is an iterative process that helps to produce more realistic models.

Validating traffic simulation models requires comparing traffic estimated by the model to observed traffic on the roadway and transit systems. Initial comparisons are for trip interchanges between quadrants, sectors, or other large areas of interest. The next step is to compare traffic estimated by the models to traffic counts, including transit ridership, crossing contrived barriers in the study area. These are typically called screenlines, cutlines, and cordon lines and may be imaginary or actual physical barriers. Cordon lines surround particular areas such as the central business district or other major activity centers. Transit ridership estimates are commonly validated by comparing them to actual patronage crossing cordon lines around the central business district.

Three sources of error can cause weak correlation during calibration: input error, model error, and parameter error. In general, input error and parameter error can be adjusted easily by the user. Model error however is caused by the methodology used in the model and may not be as easy to fix. Simulation models are typically built using several different modeling theories that can produce conflicting results. Some models are more generalized while others are more detailed. If model error occurs as a result of this, in may be necessary to adjust the model methodology to make results more consistent.

In order to produce good models that can be used to produce realistic results, these are the necessary steps that need to be taken in order to ensure that simulation models are functioning properly. Simulation models can be used as a tool to verify engineering theories but are only valid, if calibrated properly. Once satisfactory estimates of the parameters for all models have been obtained, the models must be checked to assure that they adequately perform the functions for which they are intended. The validation process establishes the credibility of the model by demonstrating its ability to replicate actual traffic patterns. The importance of model validation underscores the need for careful planning, thoroughness and accuracy of the input data collection program that has this purpose. Efforts should be made to ensure collected data is consistent with expected values. For example in traffic analysis, it is typically common for a traffic engineer to perform a site visit to verify traffic counts and become familiar with traffic patterns in the area. The resulting models and forecasts will be no better than the data used for model estimation and validation.

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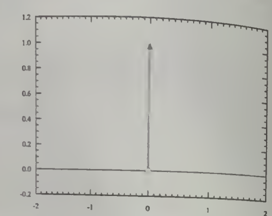
Dirac delta function

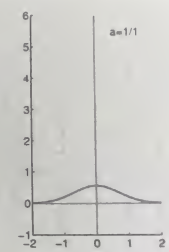


Schematic representation of the Dirac delta function by a line surmounted by an arrow. The height of the arrow is usually used to specify the value of any multiplicative constant, which will give the area under the function. The other convention is to write the area next to the arrowhead.

The **Dirac delta function**, or **δ function**, is (informally) a generalized function depending on a real parameter such that it is zero for all values of the parameter except when the parameter is zero, and its integral over the parameter from $-\infty$ to ∞ is equal to one.[1][2] It was introduced by theoretical physicist Paul Dirac. In the context of signal processing it is often referred to as the **unit impulse function**. It is a continuous analog of the Kronecker delta function which is usually defined on a finite domain, and takes values 0 and 1.

From a purely mathematical viewpoint, the Dirac delta is not strictly a function, because any extended-real function that is equal to zero everywhere but a single point must have total integral zero.[3] While for many

purposes the Dirac delta can be manipulated as a function, formally it can be defined as a distribution that is also a measure. In many applications, the Dirac delta is regarded as a kind of limit (a weak limit) of a sequence of functions having a tall spike at the origin. The approximating functions of the sequence are thus "approximate" or "nascent" delta functions.



The Dirac delta function as the limit (in the sense of distributions) of the sequence of Gaussians as

Overview

The graph of the delta function is usually thought of as following the whole x-axis and the positive y-axis. (This informal picture can sometimes be misleading, for example in the limiting case of the sinc function.)

Despite its name, the delta function is not truly a function, at least not a usual one with domain in reals. For example, the objects $f(x) = \delta(x)$ and $g(x) = 0$ are equal everywhere except at $x = 0$ yet have integrals that are different. According to Lebesgue integration theory, if f and g are functions such that $f = g$ almost everywhere, then f is integrable if and only if g is integrable and the integrals of f and g are identical. Rigorous treatment of the Dirac delta requires measure theory, the theory of distributions, or a hyperreal framework.

The Dirac delta is used to model a tall narrow spike function (an *impulse*), and other similar abstractions such as a point charge, point mass or electron point. For example, to calculate the dynamics of a baseball being hit by a bat, one can approximate the force of the bat hitting the baseball by a delta function. In doing so, one not only simplifies the equations, but one also is able to calculate the motion of

the baseball by only considering the total impulse of the bat against the ball rather than requiring knowledge of the details of how the bat transferred energy to the ball.

In applied mathematics, the delta function is often manipulated as a kind of limit (a weak limit) of a sequence of functions, each member of which has a tall spike at the origin: for example, a sequence of Gaussian distributions centered at the origin with variance tending to zero.

An infinitesimal formula for an infinitely tall, unit impulse delta function (infinitesimal version of Cauchy distribution) explicitly appears in an 1827 text of Augustin Louis Cauchy.[4] Siméon Denis Poisson considered the issue in connection with the study of wave propagation as did Gustav Kirchhoff somewhat later. Kirchhoff and Hermann von Helmholtz also introduced the unit impulse as a limit of Gaussians, which also corresponded to Lord Kelvin's notion of a point heat source. At the end of the 19th century, Oliver Heaviside used formal Fourier series to manipulate the unit impulse.[5] The Dirac delta function as such was introduced as a "convenient notation" by Paul Dirac in his influential 1927 book *Principles of Quantum Mechanics*.[6] He called it the "delta function" since he used it as a continuous analogue of the discrete Kronecker delta.

Definitions

The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

and which is also constrained to satisfy the identity

[7]

This is merely a heuristic characterization. The Dirac delta is not a true function, as no function has the above properties.[6] Moreover there exist descriptions of the delta function which differ from the above conceptualization. For example, sinc(x/a)/a becomes the delta function in the limit as $a \to 0$,[8] yet this function does not approach zero for values of x outside the origin, rather it oscillates between $1/x$ and $-1/x$ more and more rapidly as a approaches zero.

The Dirac delta function can be rigorously defined either as a distribution or as a measure.

As a measure

One way to rigorously define the delta function is as a measure, which accepts as an argument a subset A of the real line **R**, and returns $\delta(A) = 1$ if $0 \in A$, and $\delta(A) = 0$ otherwise.[9] If the delta function is conceptualized as modeling an idealized point mass at 0, then $\delta(A)$ represents the mass contained in the set A. One may then define the integral against δ as the integral of a function against this mass distribution. Formally, the Lebesgue integral provides the necessary analytic device. The Lebesgue integral with respect to the measure δ satisfies

for all continuous compactly supported functions f. The measure δ is not absolutely continuous with respect to the Lebesgue measure — in fact, it is a singular measure. Consequently, the delta measure has no Radon–Nikodym derivative — no true function for which the property

holds.[10] As a result, the latter notation is a convenient abuse of notation, and not a standard (Riemann or Lebesgue) integral.

As a probability measure on **R**, the delta measure is characterized by its cumulative distribution function, which is the unit step function[11]

This means that $H(x)$ is the integral of the cumulative indicator function $\mathbf{1}_{(-\infty, x]}$ with respect to the measure δ; to wit,

Thus in particular the integral of the delta function against a continuous function can be properly understood as a Stieltjes integral:[12]

All higher moments of δ are zero. In particular, characteristic function and moment generating function are both equal to one.

As a distribution

In the theory of distributions a generalized function is thought of not as a function itself, but only in relation to how it affects other functions when it is "integrated" against them. In keeping with this philosophy, to define the delta function properly, it is enough to say what the "integral" of the delta function against a sufficiently "good" test function is. If the delta function is already understood as a measure, then the Lebesgue integral of a test function against that measure supplies the necessary integral.

A typical space of test functions consists of all smooth functions on **R** with compact support. As a distribution, the Dirac delta is a linear functional on the space of test functions and is defined by[13]

$$\boxed{(1)}$$

for every test function φ.

For δ to be properly a distribution, it must be "continuous" in a suitable sense. In general, for a linear functional S on the space of test functions to define a distribution, it is necessary and sufficient that, for every positive integer N there is an integer M_N and a constant C_N such that for every test function φ, one has the inequality[14]

With the δ distribution, one has such an inequality (with $C_N = 1$) with $M_N = 0$ for all N. Thus δ is a distribution of order zero. It is, furthermore, a distribution with compact support (the support being $\{0\}$).

The delta distribution can also be defined in a number of equivalent ways. For instance, it is the distributional derivative of the Heaviside step function. This means that, for every test function φ, one has

Intuitively, if integration by parts were permitted, then the latter integral should simplify to

and indeed, a form of integration by parts is permitted for the Stieltjes integral, and in that case one does have

In the context of measure theory, the Dirac measure gives rise to a distribution by integration. Conversely, equation (**1**) defines a Daniell integral on the space of all compactly supported continuous functions φ which, by the Riesz representation theorem, can be represented as the Lebesgue integral of φ with respect to some Radon measure.

Generalizations

The delta function can be defined in n-dimensional Euclidean space $\mathbf{R}^n$ as the measure such that

for every compactly supported continuous function f. As a measure, the n-dimensional delta function is the product measure of the 1-dimensional delta functions in each variable separately. Thus, formally, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, one has[15]

(2)

The delta function can also be defined in the sense of distributions exactly as above in the one-dimensional case.[16] However, despite widespread use in engineering contexts, (2) should be manipulated with care, since the product of distributions can only be defined under quite narrow circumstances.[17]

The notion of a **Dirac measure** makes sense on any set whatsoever.[9] Thus if X is a set, $x_0 \in X$ is a marked point, and Σ is any sigma algebra of subsets of X, then the measure defined on sets $A \in \Sigma$ by

is the delta measure or unit mass concentrated at x_0.

Another common generalization of the delta function is to a differentiable manifold where most of its properties as a distribution can also be exploited because of the differentiable structure. The delta function on a manifold M centered at the point $x_0 \in M$ is defined as the following distribution:

(3)

for all compactly supported smooth real-valued functions φ on M.[18] A common special case of this construction is when M is an open set in the Euclidean space $\mathbf{R}^n$.

On a locally compact Hausdorff space X, the Dirac delta measure concentrated at a point x is the Radon measure associated with the Daniell integral (**3**) on compactly supported continuous functions φ. At this level of generality, calculus as such is no longer possible, however a variety of techniques from abstract analysis are available. For instance, the mapping is a continuous embedding of X into the space of finite Radon measures on X, equipped with its vague topology. Moreover, the convex hull of the image of X under this embedding is dense in the space of probability measures on X.[19]

Properties

Scaling and symmetry

The delta function satisfies the following scaling property for a non-zero scalar α:[20]

and so

(4)

In particular, the delta function is an even distribution, in the sense that which is homogeneous of degree −1.

Algebraic properties

The distributional product of δ with x is equal to zero:

Conversely, if $xf(x) = xg(x)$, where f and g are distributions, then

for some constant c.

Translation

The integral of the time-delayed Dirac delta is given by:

This is sometimes referred to as the *sifting property*[21] or the *sampling property*. The delta function is said to "sift out" the value at .

It follows that the effect of convolving a function $f(t)$ with the time-delayed Dirac delta is to time-delay $f(t)$ by the same amount:

(using (4):)

This holds under the precise condition that f be a tempered distribution (see the discussion of the Fourier transform below). As a special case, for instance, we have the identity (understood in the distribution sense)

Composition with a function

More generally, the delta distribution may be composed with a smooth function $g(x)$ in such a way that the familiar change of variables formula holds, that

provided that g is a continuously differentiable function with g' nowhere zero.[22] That is, there is a unique way to assign meaning to the distribution so that this identity holds for all compactly supported test functions f. This distribution satisfies $\delta(g(x)) = 0$ if g is nowhere zero, and otherwise if g has a real root at x_0, then

It is natural therefore to *define* the composition $\delta(g(x))$ for continuously differentiable functions g by

where the sum extends over all roots of $g(x)$, which are assumed to be simple.[22] Thus, for example

In the integral form the generalized scaling property may be written as

Properties in n dimensions

The delta distribution in an n-dimensional space satisfies the following scaling property instead:

so that δ is a homogeneous distribution of degree $-n$. Under any reflection or rotation ρ, the delta function is invariant:

As in the one-variable case, it is possible to define the composition of δ with a bi-Lipschitz function[23] $g : \mathbf{R}^n \to \mathbf{R}^n$ uniquely so that the identity

for all compactly supported functions f.

Using the coarea formula from geometric measure theory, one can also define the composition of the delta function with a submersion from one Euclidean space to another one of different dimension; the result is a type of current. In the special case of a continuously differentiable function $g : \mathbf{R}^n \to \mathbf{R}$ such that the gradient of g is nowhere zero, the following identity holds[24]

where the integral on the right is over $g^{-1}(0)$, the $n-1$ dimensional surface defined by $g(\mathbf{x}) = 0$ with respect to the Minkowski content measure. This is known as a simple layer integral.

Fourier transform

The delta function is a tempered distribution, and therefore it has a well-defined Fourier transform. Formally, one finds[25]

Properly speaking, the Fourier transform of a distribution is defined by imposing self-adjointness of the Fourier transform under the duality pairing of tempered distributions with Schwartz functions. Thus is defined as the unique tempered distribution satisfying

for all Schwartz functions φ. And indeed it follows from this that

As a result of this identity, the convolution of the delta function with any other tempered distribution S is simply S:

That is to say that δ is an identity element for the convolution on tempered distributions, and in fact the space of compactly supported distributions under convolution is an associative algebra with identity the delta function. This property is fundamental in signal processing, as convolution with a tempered distribution is a linear time-invariant system, and applying the linear time-invariant system measures its impulse response. The impulse response can be computed to any desired degree of accuracy by choosing a suitable approximation for δ, and once it is known, it characterizes the system completely. See *LTI system theory:Impulse response and convolution*.

The inverse Fourier transform of the tempered distribution $f(\xi) = 1$ is the delta function. Formally, this is expressed

and more rigorously, it follows since

for all Schwartz functions f.

In these terms, the delta function provides a suggestive statement of the orthogonality property of the Fourier kernel on **R**. Formally, one has

This is, of course, shorthand for the assertion that the Fourier transform of the tempered distribution

is

which again follows by imposing self-adjointness of the Fourier transform.

By analytic continuation of the Fourier transform, the Laplace transform of the delta function is found to be[26]

Distributional derivatives

The distributional derivative of the Dirac delta distribution is the distribution δ' defined on compactly supported smooth test functions φ by[27]

The first equality here is a kind of integration by parts, for if δ were a true function then

The k^{th} derivative of δ is defined similarly as the distribution given on test functions by

In particular δ is an infinitely differentiable distribution.

The first derivative of the delta function is the distributional limit of the difference quotients:[28]

More properly, one has

where τ_h is the translation operator, defined on functions by $\tau_h\varphi(x) = \varphi(x+h)$, and on a distribution S by

In the theory of electromagnetism, the first derivative of the delta function represents a point magnetic dipole situated at the origin. Accordingly it is referred to as a dipole or the doublet function.[29]

The derivative of the delta function satisfies a number of basic properties including:

Furthermore the convolution of δ' with a compactly supported smooth function f is

which follows from the properties of the distributional derivative of a convolution

Higher dimensions

More generally, on an open set U in the n-dimensional Euclidean space $\mathbf{R}^n$, the Dirac delta distribution centered at a point $a \in U$ is defined by[30]

for all $\varphi \in S(U)$, the space of all smooth compactly supported functions on U. If $\alpha = (\alpha_1, ..., \alpha_n)$ is any multi-index and ∂^α denotes the associated mixed partial derivative operator, then the α^{th} derivative $\partial^\alpha\delta_a$ of δ_a is given by[30]

That is, the α^{th} derivative of δ_a is the distribution whose value on any test function φ is the α^{th} derivative of φ at a (with the appropriate positive or negative sign).

The first partial derivatives of the delta function are thought of as double layers along the coordinate planes. More generally, the normal derivative of a simple layer supported on a surface is a double layer supported on that surface, and represents a laminar magnetic monopole. Higher derivatives of the delta function are known in physics as multipoles.

Higher derivatives enter into mathematics naturally as the building blocks for the complete structure of distributions with point support. If S is any distribution on U supported on the set $\{a\}$ consisting of a single point, then there is an integer m and coefficients c_α such that[31]

Representations of the delta function

The delta function can be viewed as the limit of a sequence of functions

where $\eta_\varepsilon(x)$ is sometimes called a **nascent delta function**. This limit is meant in a weak sense: either that

$\boxed{(5)}$

for all continuous functions f having compact support, or that this limit holds for all smooth functions f with compact support. The difference between these two slightly different modes of weak convergence is often subtle: the former is convergence in the vague topology of measures, and the latter is convergence in the sense of distributions.

Approximations to the identity

Typically a nascent delta function η_ε can be constructed in the following manner. Let η be an absolutely integrable function on **R** of total integral 1, and define

In n dimensions, one uses instead the scaling

Then a simple change of variables shows that η_ε also has integral 1.[32] One shows easily that (5) holds for all continuous compactly supported functions f, and so η_ε converges weakly to δ in the sense of measures. If the initial $\eta = \eta_1$ is itself smooth and compactly supported then the sequence is called a mollifier.

The η_ε constructed in this way are known as an **approximation to the identity**.[33] This terminology is because the space $L^1(\mathbf{R})$ of absolutely integrable functions is closed under the operation of convolution of functions: $f*g \in L^1(\mathbf{R})$ whenever f and g are in $L^1(\mathbf{R})$. However, there is no identity in $L^1(\mathbf{R})$ for the convolution product: no element h such that $f*h = f$ for all f. Nevertheless, the sequence η_ε does approximate such an identity in the sense that

This limit holds in the sense of mean convergence (convergence in L^1). Further conditions on the η_ε, for instance that it be a mollifier associated to a compactly supported function,[34] are needed to ensure pointwise convergence almost everywhere.

The standard mollifier is given by $\Psi(x/\varepsilon)/\varepsilon$ where Ψ is a suitably normalized bump function. For instance,

where

In some situations such as numerical analysis, a piecewise linear approximation to the identity is desirable. This can be obtained by taking η_1 to be a hat function. With this choice of η_1, one has

which are all continuous and compactly supported, although not smooth and so not a mollifier.

Probabilistic considerations

In the context of probability theory, it is natural to impose the additional condition that the initial η_1 in an approximation to the identity should be positive, as such a function then represents a probability distribution. Convolution with a probability distribution is sometimes favorable because it does not result in overshoot or undershoot, as the output is a convex combination of the input values, and thus falls between the maximum and minimum of the input function. Taking η_1 to be any probability distribution at all, and letting $\eta_\varepsilon(x) = \eta_1(x/\varepsilon)/\varepsilon$ as above will give rise to an approximation to the identity. In general this converges

more rapidly to a delta function if, in addition, η has mean 0 and has small higher moments. For instance, if η_1 is the uniform distribution on $[-1/2, 1/2]$, also known as the rectangular function, then setting with n a new parameter:[35]

Another example is with the Wigner semicircle distribution

This is continuous and compactly supported, but not a mollifier because it is not smooth.

Semigroups

Nascent delta functions often arise as convolution semigroups. This amounts to the further constraint that the convolution of η_ε with η_δ must satisfy

for all $\varepsilon, \delta > 0$. Convolution semigroups in L^1 that form a nascent delta function are always an approximation to the identity in the above sense, however the semigroup condition is quite a strong restriction.

In practice, semigroups approximating the delta function arise as fundamental solutions or Green's functions to physically motivated elliptic or parabolic partial differential equations. In the context of applied mathematics, semigroups arise as the output of a linear time-invariant system. Abstractly, if A is a linear operator acting on functions of x, then a convolution semigroup arises by solving the initial value problem

in which the limit is as usual understood in the weak sense. Setting $\eta_\varepsilon(x) = \eta(\varepsilon, x)$ gives the associated nascent delta function.

Some examples of physically important convolution semigroups arising from such a fundamental solution include the following.

The heat kernel

The heat kernel, defined by

represents the temperature in an infinite wire at time $t > 0$, if a unit of heat energy is stored at the origin of the wire at time $t = 0$. This semigroup evolves according to the one-dimensional heat equation:

In probability theory, $\eta_\varepsilon(x)$ is a normal distribution of variance ε and mean 0. It represents the probability density at time $t = \varepsilon$ of the position of a particle starting at the origin following a standard Brownian motion. In this context, the semigroup condition is then an expression of the Markov property of Brownian motion.

In higher dimensional Euclidean space $\mathbf{R}^n$, the heat kernel is

and has the same physical interpretation, *mutatis mutandis*. It also represents a nascent delta function in the sense that $\eta_\varepsilon \to \delta$ in the distribution sense as $\varepsilon \to 0$.

The Poisson kernel

The Poisson kernel

is the fundamental solution of the Laplace equation in the upper half-plane.[36] It represents the electrostatic potential in a semi-infinite plate whose potential along the edge is held at fixed at the delta function. The Poisson kernel is also closely related to the Cauchy distribution. This semigroup evolves according to the equation

where the operator is rigorously defined as the Fourier multiplier

Oscillatory integrals

In areas of physics such as wave propagation and wave mechanics, the equations involved are hyperbolic and so may have more singular solutions. As a result, the nascent delta functions that arise as fundamental solutions of the associated Cauchy problems are generally oscillatory integrals. An example, which comes from a solution of the Euler–Tricomi equation of transonic gas dynamics,[37] is the rescaled Airy function

Although using the Fourier transform, it is easy to see that this generates a semigroup in some sense, it is not absolutely integrable and so cannot define a semigroup in the above strong sense. Many nascent delta functions constructed as oscillatory integrals only converge in the sense of distributions (an example is the Dirichlet kernel below), rather than in the sense of measures.

Another example is the Cauchy problem for the wave equation in $\mathbf{R}^{1+1}$:[38]

The solution u represents the displacement from equilibrium of an infinite elastic string, with an initial disturbance at the origin.

Other approximations to the identity of this kind include the sinc function

and the Bessel function

Plane wave decomposition

One approach to the study of a linear partial differential equation

where L is a differential operator on $\mathbf{R}^n$, is to seek first a fundamental solution, which is a solution of the equation

When L is particularly simple, this problem can often be resolved using the Fourier transform directly (as in the case of the Poisson kernel and heat kernel already mentioned). For more complicated operators, it is sometimes easier first to consider an equation of the form

where h is a plane wave function, meaning that it has the form

for some vector ξ. Such an equation can be resolved (if the coefficients of L are analytic functions) by the Cauchy–Kovalevskaya theorem or (if the coefficients of L are constant) by quadrature. So, if the delta function can be decomposed into plane waves, then one can in principle solve linear partial differential equations.

Such a decomposition of the delta function into plane waves was part of a general technique first introduced essentially by Johann Radon, and then developed in this form by Fritz John (1955).[39] Choose k so that $n + k$ is an even integer, and for a real number s, put

Then δ is obtained by applying a power of the Laplacian to the integral with respect to the unit sphere measure $d\omega$ of $g(x\cdot\xi)$ for ξ in the unit sphere S^{n-1}:

The Laplacian here is interpreted as a weak derivative, so that this equation is taken to mean that, for any test function φ,

The result follows from the formula for the Newtonian potential (the fundamental solution of Poisson's equation). This is essentially a form of the inversion formula for the Radon transform, because it recovers the value of $\varphi(x)$ from its integrals over hyperplanes. For instance, if n is odd and $k = 1$, then the integral on the right hand side is

where $R\varphi(\xi,p)$ is the Radon transform of φ:

An alternative equivalent expression of the plane wave decomposition, from Gel'fand & Shilov (1966–1968, I, §3.10), is

for n even, and

for n odd.

Fourier kernels

In the study of Fourier series, a major question consists of determining whether and in what sense the Fourier series associated with a periodic function converges to the function. The n^{th} partial sum of the Fourier series of a function f of period 2π is defined by convolution (on the interval $[-\pi,\pi]$) with the Dirichlet kernel:

Thus,

where

A fundamental result of elementary Fourier series states that the Dirichlet kernel tends to the a multiple of the delta function as $N \to \infty$. This is interpreted in the distribution sense, that

for every compactly supported *smooth* function f. Thus, formally one has

on the interval $[-\pi,\pi]$.

In spite of this, the result does not hold for all compactly supported *continuous* functions: that is D_N does not converge weakly in the sense of measures. The lack of convergence of the Fourier series has led to the introduction of a variety of summability methods in order to produce convergence. The method of Cesàro summation leads to the Fejér kernel[40]

The Fejér kernels tend to the delta function in a stronger sense that[41]

for every compactly supported *continuous* function f. The implication is that the Fourier series of any continuous function is Cesàro summable to the value of the function at every point.

Hilbert space theory

The Dirac delta distribution is a densely defined unbounded linear functional on the Hilbert space L^2 of square integrable functions. Indeed, smooth compactly support functions are dense in L^2, and the action of the delta distribution on such functions is well-defined. In many applications, it is possible to identify subspaces of L^2 and to give a stronger topology on which the delta function defines a bounded linear functional.

Sobolev spaces

The Sobolev embedding theorem for Sobolev spaces on the real line $\mathbf{R}$ implies that any square-integrable function f such that

is automatically continuous, and satisfies in particular

Thus δ is a bounded linear functional on the Sobolev space H^1. Equivalently δ is an element of the continuous dual space H^{-1} of H^1. More generally, in n dimensions, one has $\delta \in H^{-s}(\mathbf{R}^n)$ provided $s > n / 2$.

Spaces of holomorphic functions

In complex analysis, the delta function enters via Cauchy's integral formula which asserts that if D is a domain in the complex plane with smooth boundary, then

for all holomorphic functions f in D that are continuous on the closure of D. As a result, the delta function δ_z is represented on this class of holomorphic functions by the Cauchy integral:

More generally, let $H^2(\partial D)$ be the Hardy space consisting of the closure in $L^2(\partial D)$ of all holomorphic functions in D continuous up to the boundary of D. Then functions in $H^2(\partial D)$ uniquely extend to holomorphic functions in D, and the Cauchy integral formula continues to hold. In particular for $z \in D$, the delta function δ_z is a continuous linear functional on $H^2(\partial D)$. This is a special case of the situation in several complex variables in which, for smooth domains D, the Szegő kernel plays the role of the Cauchy integral.

Resolutions of the identity

Given a complete orthonormal basis set of functions {} in a separable Hilbert space, for example, the normalized eigenvectors of a compact self-adjoint operator, any vector f can be expressed as:

The coefficients $\{\alpha_n\}$ are found as:

which may be represented by the notation:

a form of the bra-ket notation of Dirac.[42] Adopting this notation, the expansion of f takes the dyadic form:[43]

Letting I denote the identity operator on the Hilbert space, the expression

is called a resolution of the identity. When the Hilbert space is the space $L^2(D)$ of square-integrable functions on a domain D, the quantity:

is an integral operator, and the expression for f can be rewritten as:

The right-hand side converges to f in the L^2 sense. It need not hold in a pointwise sense, even when f is a continuous function. Nevertheless, it is common to abuse notation and write

resulting in the representation of the delta function:[44]

With a suitable rigged Hilbert space $(\Phi, L^2(D), \Phi^*)$ where $\Phi \subset L^2(D)$ contains all compactly supported smooth functions, this summation may converge in Φ^*, depending on the properties of the basis φ_n. In most cases of practical interest, the orthonormal basis comes from an integral or differential operator, in which case the series converges in the distribution sense.[45]

Infinitesimal delta functions

Cauchy used an infinitesimal to write down a unit impulse, infinitely tall and narrow Dirac-type delta function satisfying in a number of articles in 1827.[46] Cauchy defined an infinitesimal in Cours d'Analyse (1827) in terms of a sequence tending to zero. Namely, such a null sequence becomes an infinitesimal in Cauchy's and Lazare Carnot's terminology.

Modern set-theoretic approaches allow one to define infinitesimals via the ultrapower construction, where a null sequence becomes an infinitesimal in the sense of an equivalence class modulo a relation defined in terms of a suitable ultrafilter. The article by Yamashita (2007) contains a bibliography on modern Dirac delta functions in the context of an infinitesimal-enriched continuum provided by the hyperreals.

Dirac comb

A so-called uniform "pulse train" of Dirac delta measures, which is known as a Dirac comb, or as the Shah distribution, creates a sampling function, often used in digital signal processing (DSP) and discrete time signal analysis. The Dirac comb is given as the infinite sum, whose limit is understood in the distribution sense,

which is a sequence of point masses at each of the integers.

Up to an overall normalizing constant, the Dirac comb is equal to its own Fourier transform. This is significant because if f is any Schwartz function, then the periodization of f is given by the convolution

In particular,

is precisely the Poisson summation formula.[47]

Sokhatsky–Weierstrass theorem

The Sokhatsky–Weierstrass theorem, important in quantum mechanics, relates the delta function to the distribution p.v.$1/x$, the Cauchy principal value of the function $1/x$, defined by

Sokhatsky's formula states that[48]

Here the limit is understood in the distribution sense, that for all compactly supported smooth functions f,

Relationship to the Kronecker delta

The Kronecker delta is the quantity defined by

for all integers i, j. This function then satisfies the following analog of the sifting property: if be any doubly infinite sequence, then

Similarly, for any real or complex valued continuous function on , the Dirac delta satisfies the sifting property

This exhibits the Kronecker delta function as a discrete analog of the Dirac delta function.[?]

Applications to probability theory

In probability theory and statistics, the Dirac delta function is often used to represent a discrete distribution, or a partially discrete, partially continuous distribution, using a probability density function (which is normally used to represent fully continuous distributions). For example, the probability density function of a discrete distribution consisting of

points , with corresponding probabilities , can be written as

As another example, consider a distribution which 6/10 of the time returns a standard normal distribution, and 4/10 of the time returns exactly the value 3.5 (i.e. a partly continuous, partly discrete mixture distribution). The density function of this distribution can be written as

The delta function is also used in a completely different way to represent the local time of a diffusion process (like Brownian motion). The local time of a stochastic process $B(t)$ is given by

and represents the amount of time that the process spends at the point x in the range of the process. More precisely, in one dimension this integral can be written

where is the indicator function of the interval $[x-\varepsilon, x+\varepsilon]$.

Application to quantum mechanics

We give an example of how the delta function is expedient in quantum mechanics. The wave function of a particle gives the probability amplitude of finding a particle within a given region of space. Wave functions are assumed to be elements of the Hilbert space L^2 of square-integrable functions, and the total probability of finding a particle within a given interval is the integral of the magnitude of the wave function squared over the interval. A set $\{\varphi_n\}$ of wave functions is orthonormal if they are normalized by

where δ here refers to the Kronecker delta. A set of orthonormal wave functions is complete in the space of square-integrable functions if any wave function ψ can be expressed as a combination of the φ_n:

with . Complete orthonormal systems of wave functions appear naturally as the eigenfunctions of the Hamiltonian (of a bound system) in quantum mechanics that measures the energy levels, which are called the eigenvalues. The set of eigenvalues, in this case, is known as the spectrum of the Hamiltonian. In bra-ket notation, as above, this equality implies the resolution of the identity:

Here the eigenvalues are assumed to be discrete, but the set of eigenvalues of an observable may be continuous rather than discrete. An example is the position observable, $Q\psi(x) = x\psi(x)$. The spectrum of the position (in one dimension) is the entire real line, and is called a continuous spectrum. However, unlike the Hamiltonian, the position operator lacks proper eigenfunctions. The conventional way to overcome this shortcoming is to widen the class of available functions by allowing distributions as well: that is, to replace the Hilbert space of quantum mechanics by an appropriate rigged Hilbert space.[50] In this context, the position operator has a complete set of eigen-distributions,

labeled by the points y of the real line, given by

The eigenfunctions of position are denoted by in Dirac notation, and are known as position eigenstates.

Similar considerations apply to the eigenstates of the momentum operator, or indeed any other self-adjoint unbounded operator P on the Hilbert space, provided the spectrum of P is continuous and there are no degenerate eigenvalues. In that case, there is a set Ω of real numbers (the spectrum), and a collection φ_y of distributions indexed by the elements of Ω, such that

That is, φ_y are the eigenvectors of P. If the eigenvectors are normalized so that

in the distribution sense, then for any test function ψ,

where

That is, as in the discrete case, there is a resolution of the identity

where the operator-valued integral is again understood in the weak sense. If the spectrum of P has both continuous and discrete parts, then the resolution of the identity involves a summation over the discrete spectrum *and* an integral over the continuous spectrum.

The delta function also has many more specialized applications in quantum mechanics, such as the delta potential models for a single and double potential well.

Application to structural mechanics

The delta function can be used in structural mechanics to describe transient loads or point loads acting on structures. The governing equation of a simple mass-spring system excited by a sudden force impulse at time can be written

where is the mass, the deflection and the spring constant.

As another example, the equation governing the static deflection of a slender beam is, according to Euler-Bernoulli theory,

where is the bending stiffness of the beam, the deflection, the spatial coordinate and the load distribution. If a beam is loaded by a point force at , the load distribution is written

As integration of the delta function results in the Heaviside step function, it follows that the static deflection of a slender beam subject to multiple point loads is described by a set of piecewise polynomials.

Also a point moment acting on a beam can be described by delta functions. Consider two opposing point forces at a distance apart. They then produce a moment acting on the beam. Now, let the distance

approach the limit zero, while is kept constant. The load distribution, assuming a clockwise moment acting at , is written

Point moments can thus be represented by the derivative of the delta function. Integration of the beam equation again results in piecewise polynomial deflection.

Notes

[1] Dirac 1958, §15 The δ function, p. 58
[2] Gel'fand & Shilov 1968, Volume I, §§1.1, 1.3
[3] Vladimirov 1971, §5.1
[4] Laugwitz 1989, p. 230
[5] A more complete historical account can be found in van der Pol & Bremmer 1987, §V.4.
[6] Dirac 1958, §15
[7] Gel'fand & Shilov 1968, Volume I, §1.1, p. 1
[8] Weisstein, Eric W., " Delta Function (http://mathworld.wolfram.com/DeltaFunction.html)" from MathWorld.
[9] Rudin 1966, §1.20
[10] Hewitt & Stromberg 1963, §19.61
[11] Driggers 2003, p. 2321. See also Bracewell 1986, Chapter 5 for a different interpretation. Other conventions for the assigning the value of the Heaviside function at zero exist, and some of these are not consistent with what follows.
[12] Hewitt & Stromberg 1965, §9.19
[13] Strichartz 1994, §2.2
[14] Hörmander 1983, Theorem 2.1.5
[15] Bracewell 1986, Chapter 5
[16] Hörmander 1983, §3.1
[17] Strichartz 1994, §2.3; Hörmander 1983, §8.2
[18] Dieudonné 1972, §17.3.3
[19] Federer 1969, §2.5.19
[20] Strichartz 1994, Problem 2.6.2
[21] Weisstein, Eric W., " Sifting Property (http://mathworld.wolfram.com/SiftingProperty.html)" from MathWorld.
[22] Gel'fand & Shilov 1966–1968, Vol. 1, §II.2.5
[23] Further refinement is possible, namely to submersions, although these require a more involved change of variables formula.
[24] Hörmander 1983, §6.1
[25] In some conventions for the Fourier transform.
[26] Bracewell 1986
[27] Gel'fand & Shilov 1966, p. 26
[28] Gel'fand & Shilov 1966, §2.1
[29] Weisstein, Eric W., " Doublet Function (http://mathworld.wolfram.com/DoubletFunction.html)" from MathWorld.
[30] Hörmander 1983, p. 56
[31] Hörmander 1983, p. 56; Rudin 1991, Theorem 6.25
[32] Stein Weiss, Theorem 1.18
[33] Rudin 1991, §II.6.31
[34] More generally, one only needs $\eta = \eta_1$ to have an integrable radially symmetric decreasing rearrangement.
[35] Aratyn & Rasinariu 2006, §5.3.1 Dirac delta function, p. 314
[36] Stein & Weiss 1971, §I.1

[37] Vallée & Soares 2004, §7.2
[38] Hörmander 1983, §7.8
[39] See also Courant & Hilbert 1962, §14.
[40] Lang 1997, p. 312
[41] In the terminology of Lang (1997), the Fejér kernel is a Dirac sequence, whereas the Dirichlet kernel is not.
[42] The development of this section in bra-ket notation is found in (Levin 2002, Coordinate-space wave functions and completeness, pp.=109*ff*)
[43] Davis & Thomson 2000, Perfect operators, p.344
[44] Davis & Thomson 2000, Equation 8.9.11, p. 344
[45] de la Madrid, Bohm & Gadella 2002
[46] See Laugwitz (1989).
[47] Córdoba 1988, Hörmander 1983, §7.2
[48] Vladimirov 1971, §5.7
[49] Hartmann 1997, pp. 154–155
[50] Isham 1995, §6.2

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External links

- KhanAcademy.org video lesson (http://www.khanacademy.org/ video/dirac-delta-function)
- The Dirac Delta function (http://www.physicsforums.com/ showthread.php?t=73447), a tutorial on the Dirac delta function.
- Video Lectures - Lecture 23 (http://ocw.mit.edu/courses/ mathematics/18-03-differential-equations-spring-2010/ video-lectures/lecture-23-use-with-impulse-inputs), a lecture by Arthur Mattuck.
- Dirac Delta Function (http://planetmath.org/encyclopedia/ DiracDeltaFunction.html) on PlanetMath
- The Dirac delta measure is a hyperfunction (http://www. osaka-kyoiku.ac.jp/~ashino/pdf/chinaproceedings.pdf)
- We show the existence of a unique solution and analyze a finite element approximation when the source term is a Dirac delta measure (http://www.ing-mat.udec.cl/~rodolfo/Papers/BGR-3.pdf)
- Non-Lebesgue measures on R. Lebesgue-Stieltjes measure, Dirac delta measure. (http://www.mathematik.uni-muenchen.de/ ~lerdos/WS04/FA/content.html)

Lagrangian and Eulerian specification of the flow field

In fluid dynamics and finite-deformation plasticity the **Lagrangian specification of the flow field** is a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time.[1][2] Plotting the position of an individual parcel through time gives the pathline of the parcel. This can be visualized as sitting in a boat and drifting down a river.

The **Eulerian specification of the flow field** is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows as time passes.[1][2] This can be visualized by sitting on the bank of a river and watching the water pass the fixed location.

The Lagrangian and Eulerian specifications of the flow field are sometimes loosely denoted as the **Lagrangian and Eulerian frame of reference**. However, in general both the Lagrangian and Eulerian specification of the flow field can be applied in any observer's frame of reference, and in any coordinate system used within the chosen frame of reference.

Description

In the Eulerian specification of the flow field, the flow quantities are depicted as a function of fixed position $\mathbf{x}$ and time t. Specifically, the flow velocity is described as $\mathbf{u}(\mathbf{x},t)$. On the other hand, in the Lagrangian specification, all fluid parcels are labelled by some vector field $\mathbf{a}$, with $\mathbf{a}$ time-independent for each fluid parcel. Often, $\mathbf{a}$ is chosen to be the center of mass of the parcels at some initial time t_0. It is chosen in this particular manner to account for the possible changes of the shape over time. Therefore the center of mass is a good parametrization of the velocity $\mathbf{v}$ of the parcel. [1] In the Lagrangian description, the flow velocity $\mathbf{v}(\mathbf{a},t)$ is related to the position $\mathbf{X}(\mathbf{a},t)$ of the fluid parcels by:[2]

Consequently, $\mathbf{u}$ and $\mathbf{v}$ are related through

Within a chosen coordinate system, $\mathbf{a}$ and $\mathbf{x}$ are referred to as the **Lagrangian coordinates** and **Eulerian coordinates** of the flow.

The Lagrangian and Eulerian specifications of the kinematics and dynamics of the flow field are related by the substantial derivative (also called the Lagrangian derivative, convective derivative, material derivative, or particle derivative):[1]

This tells us that the total rate of change of some vector function $\mathbf{F}$ as the fluid parcels moves through a flow field described by its Eulerian

... is equal to the sum of the local rate of change ... of change of F.

Notes

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